January 2002

Combinatorics Exam

SHOW ALL WORK TO RECEIVE CREDIT!!

1. Find an explicit formula for a_k if $a_0 = 1$ and $a_{k+1} = 3a_k + 2$.

2. Let $a_0 = 1$. Find the unique sequence of real numbers satisfying

$$\sum_{i=0}^{n} a_i a_{n-i} = 1$$

for all natural numbers n.

3. A dealership has n cars. An employee with a sense of humor takes all n keys, puts one of them in each car at random, then locks the doors of all cars. When the owner of the dealership discovers the problem, he calls a locksmith. He tells him to break in to a car, then use the key found in that car to open another, and so on. If and when the keys already recovered by this procedure cannot open any new cars, the locksmith is to break in to another car. This algorithm goes on until all cars are open. What is the probability that the locksmith will only have to break in to one car ?

4. We have n distinct cards. We want to split their set into nonempty subsets so that each of them contains an even number of cards. Then we want to order the cards within each subgroup. Finally, we want to order these subgroups into a line. Find an explicit formula for the number of ways g_n we can do this.

5. How many permutations of length 8 are there with descent set $\{3,6\}$.

6. Let P_n be the poset of all parking functions of size n, with the coordinate-wise ordering. That is, $x \leq y$ if $x(i) \leq y(i)$ for all i. Add a maximum element $\hat{1}$ to the top of P, to get the poset P'. Is P' a lattice?

7. Is it true that a finite poset has as many ideals as antichains?

8. Let $f_k(n)$ be the number of permutations of length n having k descents. Prove that for any fixed k, $f_k(n)$ is a p-recursive function of n.

9. Four friends, A, B, C, and D organize a long jump competition every day until the final order of the four of them will be the same on two different days. At most how long will they have to wait for that to happen? (Each jump is measured in centimeters, so all kinds of ties, twofold, threefold, fourfold, are possible).

10. Prove that for any positive integer n, we have

$$\mu_{NC_n}(\hat{0}, \hat{1}) = (-1)^{n-1} c_n.$$

Here c_n is the nth Catalan number, and NC_n is the lattice of all noncrossing partitions of n.