Combinatorics PhD Examination

August 1995

1. Every simple planar graph has a vertex of degree at most 5.

2. (a) State the max-flow min-cut theorem for networks

   (b) Show that the max-flow min-cut theorem implies the following version of Menger's Theorem.

   The maximum number of edge disjoint directed paths between two given points s, t of a directed graph equals the minimum number of edges whose removal separates s from t.

3. We have an unlimited supply of bricks of length 1, each painted red, white or blue, and bricks of length 2, each painted green or yellow. Explicitly find a_n = the number of ways to choose a sequence of bricks of total length n.

4. State and prove Dilworth's Theorem for a finite poset.

5. Each of n gentlemen checks both his hat and his umbrella at a restaurant. Both the hats and the umbrellas are returned randomly. What is the probability that no man gets back both his hat and his umbrella?

6. Prove that
$$\sum_{i=0}^{m} \binom{s}{i}\binom{t}{k-i} = \binom{s+t}{k}.$$

7. 6 identical black balls and $n-6$ identical white balls are arranged in a row. How many ways are there to do this so that no 3 consecutive balls are black?

8. Define a q-Hamming code C as one whose parity-check matrix H has as its columns one non-zero vector from each 1-dimensional subspace of the vector space $GF(q)^r$.

   (a) What is the length of C?

   (b) What is the minimum distance in C?

   (c) Prove that C is perfect.

9. (a) State the Bruck-Ryser-Chowla theorem.

   (b) Use this theorem to prove that a projective plane of order 6 cannot exist.