

COMBINATORICS PHD EXAMINATION

May 24, 1995

1. (a) Let A_1, A_2, \dots, A_k be subsets of $[n]$ such that $A_i \cap A_j \neq \emptyset$, for all i, j . Show that

$$k \leq 2^{n-1}.$$

Is this inequality best possible?

- (b) How many pairs (A, B) of subsets of $[n]$ are there such that $A \cap B = \emptyset$?

2. Use inclusion-exclusion to determine the number of monic polynomials of degree n with no roots in $\mathbb{Z}_p[x]$.

3. (a) Prove that the number of partitions of n into at most k parts equals the number of partitions of n into parts of size of most k .

- (b) Prove that the number of partitions of n into odd parts is equal to the number of partitions of n into unequal parts. (Hint: use generating functions).

4. Prove that the number of k -dimensional subspaces of the n dimensional vector space over \mathbb{F}_q is

$$\binom{n}{k}_q = \frac{(q^n - 1) \dots (q^{n-k+1} - 1)}{(q^k - 1) \dots (q - 1)}$$

5. Prove that K_{33} is not planar.

6. Prove that every regular bipartite graph has a perfect matching. Show that this is not true in general for regular graphs of even order.

7. Prove that in a binary self dual code, either all the vectors have weight divisible by 4 or half have even weight not divisible by 4 and half have weight divisible by 4.

8. Let H be a parity check matrix for a binary (n, k, d) -code C with $n \geq 4$. If the columns of H are distinct and all have odd weight, then $d \geq 4$.

9. (a) Show the nonexistence of a difference set with parameters $(31, 10, 3)$.

- (b) Show the nonexistence of a symmetric design with parameters $(29, 8, 2)$. (Hint: consider the appropriate equation modulo 3)

- (c) Show the nonexistence of a Steiner system $S(3, 6, 11)$.

- (d) Find a $(13, 4, 1)$ difference set D . The set of translates of D form the blocks of a design. Find the parameters b, v, k, r, λ .

10. Let A be an affine plane of order n . Write $B_1 \sim B_2$ if lines B_1 and B_2 are the same or have no points in common. Show that \sim is an equivalence relation. Show that there exists a projective plane π such that A is obtained from π by deleting one line and all the points on that line.