Combinatorics Ph.D. Exam (Partial Re-examination)
September 23, 1991

Do 4 out of 5 problems.

1. A <u>$k - arc$</u> of a projective plane Π is a set of k points, no three of which are collinear. Prove that every 4-arc in the plane $PG(2,4)$ lies in exactly two 5-arcs.

2. Use (without proof) the assertion of problem #1 to prove that the 6-arcs of $\Pi = PG(2,4)$ form a block design Σ on the point set of Π, with $v = 21$ and $k = 6$. Compute λ.

3. State the Hall Multiplier Theorem, and use it to find a (91, 10, 1) difference set.

4. <u>HC</u> is the problem of deciding whether a graph has a Hamiltonian circuit or not, and this problem is known to be NP-complete. Use this fact to prove that <u>HP</u> is also NP-complete, where <u>HP</u> is the problem of deciding whether a graph has a Hamiltonian path (with unspecified initial and terminal vertices).

5. Let E be a finite set and P a partition of E. Call a subset I of E independent ($I \in \mathcal{I}$) if no two elements of I are in the same block of P.
   1. Prove that $(E, \mathcal{I})$ is a matroid (called a partition matroid).
   2. For a bipartite graph B with edge set E let $\mathcal{M} \subseteq 2^E$ denote the set of matchings on B. Show that $(E, \mathcal{M})$ is not a matroid.
   3. Prove that $\mathcal{M}$ is the intersection of two partition matroids. In other words there are partition matroids $(E, \mathcal{I}_1)$ and $(E, \mathcal{I}_2)$ such that $\mathcal{M} = \mathcal{I}_1 \cap \mathcal{I}_2$.

Part 3 is significant because there is a general result that states that if $(E, \mathcal{M})$ is such a matroid intersection then there is a polynomial algorithm to find a maximum cardinality independent set.