Combinatorics PhD Exam
May 20, 1991

Do 9 out of 12 problems. Show all of your work.

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1. How may k-tuples $(A_1, A_2, ..., A_k)$ of nested subsets of an n-set are there, where $A_1 \subseteq A_2 \subseteq ... \subseteq A_k$.

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2. How many words of length n are there in the alphabet (a,b,c) with no two adjacent a's? With no two adjacent letters equal?

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3. Find the generating function for the sequence $\{a_n\}$ given by the recurrence

$$a_n = a_{n-1} + 6a_{n-2} + 2^n$$

where $a_0 = 1$ and $a_1 = 3$.

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4. Bipartite graphs:

(a) Determine for which values of m and n the complete bipartite graph K_{mn} is (1) planar; (2) Eulerian; (3) Hamiltonian.

(b) Prove: A plane graph where every face has an even number of edges must be bipartite.

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5. Prove that $\sum_{k=1}^{n} S(n,k)x(x-1)...(x-k+1) = x^n$.

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6. Use Möbius inversion to find the number of integers between 1 and n and coprime to n, given the prime factorization of n.

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7. Prove or disprove: There is a matching from V to W in a bipartite graph if, for some fixed k, there are k or more edges incident with each vertex of V and k or fewer edges incident with each vertex of W.

8. Let $A_{m,n}$ be the collection of all sets $X = \{N_1, N_2, ..., N_k\}$ where k is a non-negative integer, and for each $i \leq k$, N_i is a positive integer divisor of $N = 2^m 3^n$, and where $N_i | N_j$ only if $i = j$. Define a partial order on $A_{m,n}$, where $X = \{N_1, N_2, ..., N_k\}$, $X' = \{N_1', N_2', ..., N_l'\}$, by $X \leq X'$ if and only if $N_i \in X \Rightarrow$ there exists $N_j' \in X'$ with $N_i | N_j'$. Prove that $A_{m,n}$ is a distributive lattice, and determine its poset of join-irreducible elements.

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9. Let I be a collection of subsets of E, where E is finite, and suppose that $I \neq \emptyset$ and that $Y \in I,\ X \subseteq Y \Rightarrow X \in I$. Let $c : E \to \mathbf{R}^+$ assign a positive real number to each element of E. Consider the optimization problem:

Find $\max\limits_{X \in I} \sum\limits_{x \in X} c(x)$.

Prove that the greedy algorithm solves this problem if and only if I is the collection of independent sets of some matroid on E.

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10. State the MacWilliams identity, and use it to compute the weight polynomial of the (15,11) Hamming binary code.

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11. A (v, k, λ)-design Σ is an incidence structure consisting of a set S of v points and a collection of k-subsets of S called blocks, such that every pair of points of S is contained in precisely λ blocks.

Prove that, in the range $30 \leq v \leq 45$, a $(v, 6, 1)$-design exists if and only if $v = 31$.

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12. Use the Hall Multiplier Theorem

(i) to prove the non-existence of a cyclic (31, 10, 3) difference set.

(ii) to find a cyclic (31, 6, 1) difference set.