PhD Analysis Examination
May 2016

Do THREE problems from Part A and THREE problems from Part B. Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

Part A

---

1. Consider Lebesgue measure on the real line $\mathbb{R}$. Give an example for each of the following, if possible. If not possible, give a brief explanation.

   A sequence (f_n) in $L^1(\mathbb{R})$ converging to an f in $L^1(\mathbb{R})$...

   a) ...in the L^1 norm but not in measure,

   b) ...in measure but not in the L^1 norm,

   c) ...in the L^1 norm but not a.e.,

   d) ...a.e. but not in measure.

2. Let ν be a signed measure on $(X, \mathscr{M})$.

   a) Define what it means for a set to be *positive, negative* or *null* for ν, and state the Hahn decomposition theorem.

   b) Prove there exist unique positive measures ν^+, ν^- on $\mathscr{M}$ so that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.

3. Let μ be a finite, regular Borel measure on $[0,1]$. Suppose that

$$\int_0^1 x^n \, d\mu = 0 \quad \text{for } n = 0, 1, 2, \ldots$$

   Prove that $\mu = 0$.

4. Let $(X, \mathscr{M}, \mu)$ be a σ-finite measure space. Let $\mathscr{N}$ be a sub-σ algebra of $\mathscr{M}$ and let ν be the restriction of μ to $\mathscr{N}$. Prove that for each $f \in L^1(\mu)$, there exists a $g \in L^1(\nu)$ such that

$$\int_E f \, d\mu = \int_E g \, d\nu$$

   for all $E \in \mathscr{N}$.

Part B

---

1. Let $\mathcal{X}$ be a normed vector space. Say a sequence (x_n) from $\mathcal{X}$ converges *weakly* to $x \in \mathcal{X}$ if $f(x_n) \to f(x)$ for all $f \in \mathcal{X}^*$. Prove that if $\mathcal{M}$ is a norm closed subspace of $\mathcal{X}$, and (x_n) is a sequence in $\mathcal{M}$ converging weakly to $x \in \mathcal{X}$, then $x \in \mathcal{M}$.

2. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M}, \mathcal{N}$ closed subspaces of $\mathcal{H}$. Prove that if $\mathcal{M} \perp \mathcal{N}$, then $\mathcal{M} + \mathcal{N}$ is closed.

3. Fix $1 < p < \infty$, let μ be a σ-finite measure, and let (f_n) be a sequence in $L^p(\mu)$. Suppose that there exists a function $f \in L^p(\mu)$ such that

$$\int f_n g \, d\mu \to \int f g \, d\mu$$

   for every $g \in L^q(\mu)$, where $\frac{1}{p} + \frac{1}{q} = 1$.

   a) Prove that $\sup_n \{\|f_n\|_p\} < +\infty$.
   b) Prove that $\|f\|_p \leq \limsup \|f_n\|_p$.

4. Let (φ_n) be a sequence in $L^1(\mathbb{R})$ with the following properties:

   i) $\varphi_n \geq 0$ for all n,
   ii) $\int_{\mathbb{R}} \varphi_n(x) \, dx = 1$ for all n, and
   iii) for every $\delta > 0$, $\lim_{n\to\infty} \int_{|x|>\delta} \varphi_n(x) \, dx = 0$.

   Prove that $f * \varphi_n \to f$ in $L^1(\mathbb{R})$.