PhD Analysis Exam, Spring 2015

Do six of eight. Answer each problem on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

1. Let X and Y be topological spaces. Prove, if $f : X \to Y$ is continuous, then f is $\mathscr{B}_X$-$\mathscr{B}_Y$ measurable (that is, measurable with respect to the Borel σ-algebras on X and Y respectively).

2. (Short Answer.) Do two of three.

   (i) Give an example, if possible, of a closed subset F of $\mathbb{R}$ of positive measure which contains no nontrivial open interval.

   (ii) Give an example, if possible, which shows that the σ-finite hypothesis is needed in Tonelli's Theorem.

   (iii) Determine the limit of the sequence,
   $$\int_1^\infty \frac{\sin(nx)}{x^n}\, dx.$$

3. Let $(X, \mathscr{M}, \mu)$ be a measure space and $(f_n)_{n=1}^\infty$ be a sequence from $L^1(\mu)$. Prove, if there is a $g \in L^1(\mu)$ such that $g(x) \geq |f_n(x)|$ for all n and $x \in X$, then
$$\lim_{\delta \to 0} \sup_n \int_{|f_n|<\delta} |f_n|\, d\mu = 0.$$

4. Prove, if $E \subset \mathbb{R}$ is a Lebesgue measurable set of positive measure, then $E - E = \{x - y : x, y \in E\}$ contains a nontrivial open interval.

5. Let $(X, \mathscr{M}, \mu)$ be a measure space and $1 \leq p < q < \infty$. Prove, $L^q(\mu) \subset L^p(\mu)$ if and only if there is a (real) constant K such that for each $E \in \mathscr{M}$ either $\mu(E) \leq K$ or $\mu(E) = \infty$.

6. Prove, if X is a Banach space and M is a finite dimensional subspace of X, then there is a closed subspace N of X such that $M \cap N = (0)$ and $M + N = X$.

7. Suppose $\|\cdot\|$ and $\|\cdot\|_*$ are both norms on a vector space X and $\|x\| \leq \|x\|_*$ for all $x \in X$. Show, if both $(X, \|\cdot\|)$ and $(X, \|\cdot\|_*)$ are Banach spaces, then these norms are equivalent.

8. (Short Answer. Do two of three)

   (a) Explain why c_{00} can not be a Banach space.

   (b) Explain why there is no bounded one-one onto linear map from c_0 to ℓ^∞.

   (c) Explain what is meant by the Fourier transform of a function $f \in L^2(\mathbb{R})$.