PhD Analysis Examination
May 2014

You must answer SIX questions. Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

1. Let $(X, \mathscr{M}, \mu)$ be a measure space and $f \in L^1(\mu)$. Prove that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $\mu(E) < \delta$, then $\int_E |f| \, d\mu < \epsilon$.

2. State the Hahn decomposition theorem for signed measures. Prove that if ρ is a signed measure on a measurable space $(X, \mathscr{M})$, then there exist unique positive measures ρ_+, ρ_- such that $\rho_+ \perp \rho_-$ and $\rho = \rho_+ - \rho_-$.

3. State the Fubini-Tonelli theorem and give a sketch of its proof.

4. Let $C^1[0,1]$ denote the set of all continuous (real-valued) functions f on $[0,1]$ such that f is differentiable in $(0,1)$ and f' extends continuously to $[0,1]$. Prove that $C^1[0,1]$ is a Banach space under the norm

$$\|f\| = \|f\|_\infty + \|f'\|_\infty.$$

5. Let $\mathcal{X}$ be a normed vector space (over $\mathbb{C}$) and $\mathcal{M} \subset \mathcal{X}$ a closed subspace. Prove that if $x \in \mathcal{X} \setminus \mathcal{M}$, then $\mathcal{M} + \mathbb{C}x$ is closed.

6. a) State the Closed Graph Theorem and the Banach Isomorphism Theorem. b) Assuming the Banach Isomorphism Theorem (or otherwise), prove the Closed Graph Theorem.

7. Let $\mathcal{X}$ be a Banach space and $\mathcal{Y}$ a normed vector space. Suppose that T_n is a sequence of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$. Prove that if $\lim T_n x$ exists for each $x \in \mathcal{X}$, then the mapping $Tx = \lim T_n x$ defines a bounded linear operator from $\mathcal{X}$ to $\mathcal{Y}$.

8. Let $(X, \mathscr{M}, \mu)$ be a σ-finite measure space. Prove that the simple functions that belong to $L^2(\mu)$ are dense in $L^2(\mu)$.