**Ph.D. Exam in Analysis    August 2012**

*Be sure to carefully present all work in a neat and logical fashion. Do not leave any gaps. State clearly theorems used in your proofs. Print your name on each sheet turned in.*

(1) (a) State the Vitali convergence theorem (define all terms).
(b) Use this theorem to prove the Lebesgue Dominated Convergence theorem (first state this theorem).

(2) Let $f$ be an integrable function on $(S, \Sigma, \mu)$. Let $\nu$ be the indefinite integral of $\nu$. Let $|\nu|(.)$ denote the total variation of $\nu$ defined on $\Sigma$. Show that $|\nu|(.)$ is the indefinite integral of an a.e. unique integrable function $g$. What is the relationship between $g$ and $f$? Prove.

(3) Let $(S, \Sigma, \mu)$ be a finite measure space. Suppose $x^*$ belongs to the dual space of $L^1(S, \Sigma, \mu)$. Show that there exist an a.e. unique integrable function $g$ such that

$$x^*(f) = \int_S fg\, d\mu, \text{ for each } f \in L^1(S, \Sigma, \mu).$$

[You need not prove that $g \in L^\infty$, etc].

(4) Let $\{y_n\}$ be an orthogonal sequence in a Hilbert space. Show $\sum y_n$ converges unconditionally if and only if $\sum |y_n|^2 < \infty$.

(5) Let $X$ be a Banach space with closed linear subspaces $Y$ and $Z$. Suppose each $x$ in $X$ is the unique sum $y + z$ where $y \in Y$ and $z \in Z$. Show that there is a constant $K$ such that $|y| \leq K|x|$ and $|z| \leq K|x|$ for each $x$ in $X$ with representation $x = y + z$.
[Hint: Use the open mapping theorem or closed graph theorem for certain maps]

(6) Let $\{x_n\}$ be a weakly convergent sequence in a normed space $X$. Show that the weak limit belongs to the norm closed span of $x_n$.