Analysis
PhD Examination
January 2012

Answer SIX questions. Write solutions in a neat and logical fashion, giving complete reasons for all steps and stating carefully any substantial theorems used.

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1. Let V be the vector space of *all* complex sequences $z = (z_n)_{n \in \mathbb{N}}$ and for each $n \in \mathbb{N}$ denote the nth coordinate map by

$$\phi_n : V \to \mathbb{C} : z \mapsto z_n.$$

Does V carry a norm relative to which the linear functional ϕ_n is bounded for *infinitely many* values of n? Explain.

2. State the Baire Category Theorem.

Decide whether the vector space c_{00} comprising all finitely-nonzero complex sequences carries a *complete* norm (by considering suitable subspaces of finite dimension, or otherwise).

3. State the Closed Graph Theorem.
Prove the Banach Isomorphism Theorem.

4. Let $1 \leqslant p < \infty$ and define norms $\|\cdot\|_p$ and $\|\cdot\|_\infty$ on $C[0,1] \ni f$ by

$$\|f\|_p = \Big(\int_0^1 |f(t)|^p \mathrm{d}t \Big)^{1/p}$$

$$\|f\|_\infty = \sup\{|f(t)| : 0 \leqslant t \leqslant 1\}$$

as usual. Show that on $C[0,1]$:
(i) $\|\cdot\|_p$ and $\|\cdot\|_\infty$ are inequivalent;
(ii) $\|\cdot\|_p$ is incomplete.

5. Let $\mathbb{H}$ be a complex Hilbert space and $\phi : \mathbb{H} \to \mathbb{C}$ a bounded linear functional. Prove that there exists a unique $u \in \mathbb{H}$ such that $\phi(v) = \langle u|v \rangle$ for each $v \in \mathbb{H}$.

6. Say what it means for one measure to be *absolutely continuous* relative to another.

Let $(\mu_n)_{n=1}^{\infty}$ be a sequence of finite measures on the σ-algebra $\mathcal{F}$. Prove that there exists a finite measure λ on $\mathcal{F}$ such that $\mu_n \ll \lambda$ for each $n \geqslant 1$.

7. Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space on which f is a measurable real-valued function. Prove that the rule

$$t \in \mathbb{R} \Rightarrow F(t) = \int_{\Omega} e^{itf(\omega)} \mathrm{d}\mu(\omega)$$

defines a continuous function $F : \mathbb{R} \to \mathbb{C}$.

8. Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space, let $g : \Omega \to \mathbb{C}$ be measurable and let $p \geqslant 1$. Show that pointwise multiplication

$$f \in L_p \Rightarrow \Phi_g(f) = gf$$

defines a bounded linear operator Φ_g on L_p iff $g \in L_\infty$. *Note*: Recall that $g \in L_\infty$ iff there exists $K \geqslant 0$ such that $\mu\{|g| > K\} = 0$.