Analysis
PhD Examination
September 2011

Answer SIX questions. Write solutions in a neat and logical fashion, giving complete reasons for all steps and stating carefully any substantial theorems used.

1. Let Y_p denote $\mathbb{R}^3$ provided with the p-norm. Are the normed spaces Y_1 and Y_∞ isometrically isomorphic? Prove or disprove.

2. Let X and Y be normed spaces, X being complete. Let $(T_n)_{n=1}^\infty$ be a sequence of bounded linear operators $X \to Y$ and assume that for each $x \in X$ the limit $\lim_{n\to\infty} T_n x =: Tx$ exists in Y. Show that $T : X \to Y$ is a bounded linear operator.

3. State the Closed Graph theorem and the Open Mapping theorem. Derive ONE of these from the Banach Isomorphism theorem.

4. Let $\mathbb{H}$ be a Hilbert space. Prove that linear maps S and T from $\mathbb{H}$ to itself that satisfy

$$(\forall x, y \in \mathbb{H}) \;\; \langle Sx|y\rangle = \langle x|Ty\rangle$$

are automatically continuous.

5. Let ϕ be a bounded linear functional on the real Hilbert space $\mathbb{H}$ and for $x \in \mathbb{H}$ define

$$f(x) = \|x\|^2 - \phi(x).$$

Prove that each nonempty closed convex set $K \subset \mathbb{H}$ contains a unique point at which the restriction $f|_K$ is minimized. *Suggestion*: represent the bounded linear functional.

6. Let f be a real-valued measurable function on the measure space $(\Omega, \mathcal{F}, \mu)$. Prove that if $A \in \mathcal{F}$ is an atom then there exists a real number a such that $f = a$ almost everywhere on A. *Note*: To say that $A \in \mathcal{F}$ is an *atom* means that $\mu(A) > 0$ and if $B \in \mathcal{F}$ is a subset of A then either $\mu(B) = 0$ or $\mu(A \setminus B) = 0$.

7. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Prove that if $1 < p < s < q < \infty$ then

$$\mathcal{L}^p(\mu) \cap \mathcal{L}^q(\mu) \subseteq \mathcal{L}^s(\mu).$$

8. State the (σ-finite) Radon-Nikodym theorem. Prove that if the σ-finite measures λ, μ, ν satisfy $\nu \ll \mu$ and $\mu \ll \lambda$ then (λ-ae)

$$\frac{\mathrm{d}\nu}{\mathrm{d}\lambda} = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}.$$