Analysis
PhD Examination
May 2011

Answer SIX questions. Write solutions in a neat and logical fashion, giving complete reasons for all steps and stating carefully any substantial theorems used.

1. (i) State the Hahn-Banach theorem.
(ii) Show that there exists an isometric linear map from each separable normed space into ℓ_∞.

2. Prove that the spaces c (of all convergent scalar sequences) and c_0 (of all null scalar sequences) are not *isometrically* isomorphic when equipped with the sup norm.

3. State the Closed Graph Theorem and the Banach Isomorphism Theorem; deduce *one* of these from the other.

4. Let the sequence $(T_n)_{n=1}^{\infty} \subset L(X, Y)$ be bounded in operator norm and assume Y complete. Prove that

$$Z = \{z \in X : (T_n z)_{n=1}^{\infty} \text{ converges}\}$$

is a closed subspace of X.

5. Let X and Y are closed subspaces of a Hilbert space. Prove that if $X \perp Y$ then $X + Y$ is closed.

6. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Define $\overline{\mathcal{F}}$ to comprise all those $A \subseteq \Omega$ for which there exist $L, U \in \mathcal{F}$ such that $L \subseteq A \subseteq U$ and $\mu(U \smallsetminus L) = 0$ and then define $\overline{\mu}(A)$ to be the common value $\mu(L) = \mu(U)$. Show that $(\Omega, \overline{\mathcal{F}}, \overline{\mu})$ is a measure space that is *complete* in the sense that each subset of a null set is null.

7. Let $(\Omega, \mathcal{F}, \mu)$ be an arbitrary measure space; let $p, q, r > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Prove that if $f \in \mathcal{L}_p(\mu)$ and $g \in \mathcal{L}_q(\mu)$ then $fg \in \mathcal{L}_r(\mu)$ with

$$\|fg\|_r \leqslant \|f\|_p \|g\|_q.$$

8. Let $1 \leqslant p < q < \infty$. Show that $\mathcal{L}_p(\mu) \nsubseteq \mathcal{L}_q(\mu)$ iff $(\Omega, \mathcal{F}, \mu)$ contains measurable sets of arbitrarily small positive measure.