PhD Analysis Examination
September 2010

Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

Attempt EIGHT problems.

1. Let $f \in L^1[0,1]$. Suppose that
$$\int_0^1 x^n f(x)\,dx = 0 \quad \text{for all } n = 0, 1, 2, \ldots$$
Prove that $f = 0$ almost everywhere. (Hint: reduce the problem to showing that if $\int_E f(x)\,dx = 0$ for every Borel E, then $f = 0$ a.e.)

2. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, and let $g : X \to \mathbb{C}$ be a measurable function. Prove that the operator $M_g : f \to gf$ is bounded on $L^p(\mu)$ $(1 \le p \le \infty)$ if and only if $g \in L^\infty(\mu)$, in which case $\|M_g\| = \|g\|_\infty$.

3. a) Define *monotone class*, and state the Monotone Class Theorem.

   b) State the Fubini-Tonelli theorem.

4. a) Let ν be a signed measure on $(X, \mathcal{M})$. Define what it means for a set to be *positive, negative* or *null* for ν, and state the Hahn decomposition theorem.

   b) Prove that if ν is a signed measure, then there exist unique positive measures ν^+, ν^- so that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.

5. Let $(k_n)_{n=1}^\infty$ be a sequence of measurable, 1-periodic functions on $\mathbb{R}$ with the following properties:

   a) $k_n \ge 0$ a.e., for all n.

   b) $\int_{-1/2}^{1/2} k_n(t)\,dt = 1$ for all n.

   c) For each $\delta > 0$, $k_n \to 0$ uniformly on the set $\delta \le |t| \le \frac{1}{2}$.

   Prove that if $f \in L^1[-\frac{1}{2}, \frac{1}{2}]$, then $f * k_n \to f$ in $L^1[-\frac{1}{2}, \frac{1}{2}]$.

   (Here $*$ denotes convolution on $[-\frac{1}{2}, \frac{1}{2}]$: $(f * g)(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t)g(x-t)\,dt$.)