PhD Analysis Examination
May 2010

Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

Do any seven problems.

1. Consider Lebesgue measure on the real line $\mathbb{R}$. Give an example for each of the following, if possible. If not possible, give a brief explanation.

   A sequence f_n in $L^1(\mathbb{R})$ converging to an f in $L^1(\mathbb{R})$...

   a) ...in the L^1 norm but not in measure,

   b) ...in measure but not in the L^1 norm,

   c) ...in the L^1 norm but not a.e.,

   d) ...a.e. but not in measure.

   Which of these answers change if we replace $\mathbb{R}$ by the interval $[0, 1]$?

2. Define the Hardy-Littlewood maximal function. State and prove the Hardy-Littlewood maximal theorem. (Begin by proving an appropriate covering lemma.)

3. a) State Egorov's theorem.

   b) Suppose $(X, \mathscr{M}, \mu)$ is a measure space with $\mu(X) < \infty$ and $f : X \to \mathbb{C}$ is measurable. Let (f_n) be a sequence of integrable functions such that $f_n \to f$ a.e. Suppose further that the sequence f_n is *uniformly integrable*, that is, for every $\epsilon > 0$ there exists a $\delta > 0$ such that if E is measurable and $\mu(E) < \delta$, then

   $$\int_E |f_n| \, d\mu < \epsilon.$$

   Prove that f is integrable and $\lim_{n\to\infty} \int_X |f_n - f| \, d\mu = 0$.

4. Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces. Suppose that (T_n) is a sequence of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$ such that $\lim T_n x$ exists for every $x \in \mathcal{X}$. Prove that

$$Tx := \lim T_n x$$

defines a bounded linear operator from $\mathcal{X}$ to $\mathcal{Y}$.

5. a) State the Lebesgue-Radon-Nikodym theorem.

   b) State and prove a version of the chain rule for Radon-Nikodym derivatives.

6. Let X be a normed vector space. Given $x \in X$, define a linear functional $\hat{x} : X^* \to \mathbb{C}$ by $\hat{x}(f) = f(x)$. Prove that the map $x \to \hat{x}$ is an isometry from X into X^{**}.

7. Let $\mathcal{H}$ be a Hilbert space and suppose $\mathcal{M}$, $\mathcal{N}$ are closed subspaces with $\mathcal{M} \perp \mathcal{N}$. Prove that $\mathcal{M} + \mathcal{N}$ is closed.

8. Suppose $(k_n)_{n=1}^{\infty}$ is a sequence of functions in $L^1(\mathbb{R})$ satisfying:

   a) $\displaystyle\int_{-\infty}^{\infty} k_n(t)\,dt = 1$ for all n,

   b) $\displaystyle\sup_n \int_{-\infty}^{\infty} |k_n(t)|\,dt < \infty$, and

   c) for each $\delta > 0$, $\displaystyle\sup_{|t|\geq\delta}\{|k_n(t)|\} \to 0$ as $n \to \infty$.

   Prove that $\displaystyle\lim_{n\to\infty} \|f - f * k_n\|_1 = 0$ for all $f \in L^1(\mathbb{R})$.

   (Here $*$ denotes convolution: $(f * g)(x) := \int_{-\infty}^{\infty} f(x-t)g(t)\,dt$.)