Analysis
PhD Examination
Sept. 2009

Do 6 out of the 7 problems

1. State and prove the Fubini Theorems.

2. State and prove the Vitali integral convergence theorem.

3. Let μ be a signed measure on the σ-algebra $\mathcal{S}$. Define positive and negative sets for μ.
   Prove that if $E \in \mathcal{S}$, then $\mu^+(E) = \sup\{\mu(F) : F \subset E, F \in \mathcal{S}\}$.

4. Let Y be an orthogonal set in a Hilbert space.
   Prove that $\sum_{y \in Y} y$ converges if and only if $\sum_{y \in Y} \|y\|^2 < \infty$.

5. Let X be a Banach space, $A \subset X$. Suppose that $\sup_{a \in A} |x^* a| < \infty$, for each $x^* \in X^*$. Prove $\sup_{a \in A} \|a\| < \infty$.

6. Let μ and ν be finite measures such that each is absolutely continuous with respect to the other.
   Prove $\dfrac{d\mu}{d\nu} = \dfrac{1}{\left(\frac{d\nu}{d\mu}\right)}$ a.e. μ.

7. Let X, Y, Z be Banach spaces and let $\mathcal{F}$ be a total family of continuous linear maps on X to Y. Let $T: Z \to X$ be a linear map such that fT is continuous for every $f \in \mathcal{F}$. Prove T is continuous. [Hint: Use closed graph Theorem]

<u>Note</u> $\mathcal{F}$ <u>total</u> means if $f(x) = 0$ for all $f \in \mathcal{F}$, then $x = 0$.