Analysis
PhD Examination
May 2009

1. State and prove the Fubini theorems.

2. (a) State the Vitali integral convergence theorem.
(b) State the Lebesgue Dominated Convergence Theorem.
(c) Use (a) to prove (b) [the convergence in measure version].

3. (a) Let $\mu : \Sigma \to [-\infty, \infty]$ be countably additive, where Σ is a σ-algebra. Define positive and negative sets for μ and state the Hahn decomposition theorem for μ.
(b) Let μ be as above. Does there exist a set $E \in \Sigma$ such that $\mu(E)$ is the minimum value for μ on Σ? Prove.

4. Suppose $\{f_n\}$ is a sequence of integrable functions with the property that $\sum \int |f_n| \mathrm{d}\mu < \infty$. Analyze the pointwise convergence of $\sum f_n(x)$, for $x \in X$.

5. Let Y be an orthonormal set in a Hilbert space and suppose $f : Y \to \mathbb{C}$ is given. State necessary and sufficient conditions for $\sum_{y \in Y} f(y)y$ to converge. Prove.

6. Let $\mathfrak{X}$ be a normed space and $\pi : \mathfrak{X} \to \mathfrak{X}^{**}$ the natural map. Under what conditions is $\pi(\mathfrak{X})$ closed in $\mathfrak{X}^{**}$ (equipped with the norm topology)? Prove.

7. (a) State the Principle of Uniform Boundedness for a family of operators.
(b) Let $\mathfrak{X}$ be a normed space. Use (a) to show that if $A \subset \mathfrak{X}$ satisfies $\sup_{a \in A} |x^* a| < \infty$ for each $x^* \in \mathfrak{X}^*$ then $\sup_{a \in A} |a| < \infty$.

8. Let $\mathfrak{X}$ be a normed space, and assume M is a closed linear subspace of $\mathfrak{X}$. Let $z \in \mathfrak{X} \setminus M$ and $S = \operatorname{span}\{z, M\}$. Show S is closed in $\mathfrak{X}$.

[Hint: Let $f : S \to \mathbb{C}$ be defined by $f(x + \alpha z) = \alpha$, for $x \in M$ and $\alpha \in \mathbb{C}$. Show $\|f\| \leqslant 1/\operatorname{dist}(z, M)$, hence $f \in S^*$.]