PhD Analysis Examination
May 2008

Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

1. Define the Hardy-Littlewood maximal function. State and prove the Hardy-Littlewood maximal theorem. (Begin by proving a suitable covering lemma.)

2. Suppose $f_n \to f$ in measure. Prove the following:

   a) If $f_n \geq 0$ for all n, then $\int f \leq \liminf \int f_n$.

   b) If $|f_n| \leq g$ for all n and $g \in L^1$, then $\int f = \lim \int f_n$.

3. Let $(X, \mathscr{M}, \mu)$ be a σ-finite measure space, $\mathscr{N}$ a sub-σ-algebra of $\mathscr{M}$, and $\nu = \mu|_{\mathscr{N}}$. Prove that if $f \in L^1(\mu)$ then there exists $g \in L^1(\nu)$ (unique modulo ν-null sets) such that

   $$\int_E f\, d\mu = \int_E g\, d\nu$$

   for all $E \in \mathscr{N}$.

4. Suppose $1 < p < \infty$, $f \in L^p(0, \infty)$, and $p^{-1} + q^{-1} = 1$. Define

   $$F(x) = \int_0^x f(t)\, dt.$$

   Show that $\dfrac{F(x)}{x^{1/q}} \to 0$ as $x \to 0$ and $x \to \infty$.

5. Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces. Suppose that (T_n) is a sequence of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$ such that $\lim T_n x$ exists for every $x \in \mathcal{X}$. Prove that

   $$Tx := \lim T_n x$$

   defines a bounded linear operator from $\mathcal{X}$ to $\mathcal{Y}$.

6. Let μ be a σ-finite positive measure and $1 \leq p \leq \infty$. Let $g \in L^\infty$. Prove that the operator

   $$Tf = gf$$

   is bounded on L^p, and $\|T\| = \|g\|_\infty$.

7. Let $\mathcal{X}$ be a normed vector space over $\mathbb{C}$. Prove that if $\mathcal{M}$ is a closed subspace of $\mathcal{X}$ and $x \in \mathcal{X} \setminus \mathcal{M}$, then the subspace $\mathcal{M} + \mathbb{C}x$ is closed.

8. Let $\mathcal{E}$ be a subset of a Hilbert space $\mathcal{H}$. Prove that $(\mathcal{E}^\perp)^\perp$ is equal to the smallest closed subspace of $\mathcal{H}$ containing $\mathcal{E}$.