Analysis
PhD Examination

August 2007

Attempt EIGHT questions

To receive credit be sure to state clearly the theorems used in solutions

① Let $(X, \mathcal{S}, \mu)$ be the following measure space: X is the set of positive integers, $\mathcal{S}$ is the power set of X and μ is the counting measure.
If f is μ-integrable, does $\sum |f(n)|$ converge? Prove.
Is the converse true? Prove.

② Let μ and ν be finite valued signed measures such that $\mu \ll \nu$ and $\nu \ll \mu$. Prove that $\left(\frac{d\mu}{d\nu}\right)\left(\frac{d\nu}{d\mu}\right) = 1$ a.e.

③ Let Σ be an algebra of subsets of a set S and $\sigma(\Sigma)$ the σ-algebra generated by Σ. State the Hahn extension theorem for a σ-finite measure μ defined on Σ.
Use a Monotone Class argument to prove that the extension is unique.

④ Let $\{x_n\}$ be a sequence of elements from a Banach space $\mathcal{X}$.
Suppose $\sum \|x_n\| < \infty$. Does $\sum x_n$ converge unconditionally? Prove

⑤ Let $\mathcal{M}$ and $\mathcal{N}$ be closed linear subspaces of a Hilbert space.
Assume $\mathcal{M} \perp \mathcal{N}$. Show $\mathcal{M} + \mathcal{N}$ is closed.

(6) Let $\mathfrak{X}$ be a Banach space and suppose $\pi : \mathfrak{X} \to \mathfrak{X}^{**}$ is the natural embedding map. Is $\pi(\mathfrak{X})$ closed in $\mathfrak{X}^{**}$? Prove

(7) Let $L^1(\mathbb{R})$ be the space of Lebesgue integrable functions on $\mathbb{R}$. Assume E is a Lebesgue measurable set. Denote by Y the set of all functions $f \in L^1(\mathbb{R})$ such that $f(t) = f(-t)$ for almost all $t \in E$. Prove Y is closed in $L^1(\mathbb{R})$.

(8) Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{D}$ a sub σ-algebra of $\mathcal{F}$. Suppose $f \in L^2(\Omega, \mathcal{F}, P)$. Show by Hilbert space theory that there exists a $\mathcal{D}$-measurable function h such that

$$\int_D f \, dP = \int_D h \, dP, \quad D \in \mathcal{D}.$$

(9) Let $\mathcal{Y}$ and $\mathcal{Z}$ be closed linear subspaces of the Banach space $\mathfrak{X}$ such that each $x \in \mathfrak{X}$ has a unique representation $x = y + z$, where $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$.
Show that there exists a constant K such that $\|y\| \le K\|x\|$ and $\|z\| \le K\|x\|$, for all $x \in \mathfrak{X}$, where $x = y + z$ is the above representation of x.