Analysis Exam

PH.D. QUALIFYING EXAM IN MEASURE AND INTEGRATION THEORY
SEPTEMBER 6, 2005

Give complete proofs and computations. Be particularly careful to indicate why the most crucial steps in your proof are true. Partial credit will be given where justified.

1) Let $\Omega \subset \mathbb{R}^n$ be open. Prove that $A \subset \mathcal{D}(\Omega)$ is bounded if and only if

$$\sup\{|\Lambda(\varphi)| \mid \varphi \in A\} < \infty,$$

for every $\Lambda \in \mathcal{D}'(\Omega)$.

2) Let μ be Lebesgue measure on $\mathbb{R}$ and f be μ-integrable on $[0,1]$. Suppose $\int_0^r f\,d\mu = 0$, for every rational $r \in [0,1]$. Give a complete characterization of f.

3) Let $\mathcal{H}$ be a Hilbert space and suppose A is a bounded operator on $\mathcal{H}$ such that the range of A is finite dimensional. Let $\{y_1, \ldots, y_n\}$ be an orthonormal basis for the range of A. Prove that there exist $x_1, \ldots, x_n \in \mathcal{H}$ such that

$$Ax = \sum_{j=1}^{n} \langle x, x_j \rangle\, y_j\,,$$

for all $x \in \mathcal{H}$.

4) Let A be a bounded operator on a Hilbert space $\mathcal{H}$ such that $A(B_1) = \{Ax \mid x \in B_1\}$ is compact, where B_1 is the closed unit ball of $\mathcal{H}$. Prove that, as a map from B_1 with the weak vector topology to $\mathcal{H}$ with the norm topology, A is continuous. (Hint: show that the weak and norm topologies on $A(B_1)$ must coincide under the given hypothesis.)

5) Let μ be a positive Radon measure on $\mathbb{R}^N$ and f be μ-integrable. Show that for each $\epsilon > 0$ there exist an upper semi-continuous function u_ϵ and a lower semi-continuous function v_ϵ such that $u_\epsilon \le f \le v_\epsilon$ and $\int (v_\epsilon - u_\epsilon)\,d\mu < \epsilon$.

6) Let μ be a finite positive Radon measure on $\mathbb{R}^N$ and $\{f_n\}_{n\in\mathbb{N}} \subset L^2(\mu)$ with $\|f_n\|_2 \le 1$. Suppose $\{f_n\}_{n\in\mathbb{N}}$ coverges μ-a.e. to f. Show that $\int |f_n - f|\,d\mu \to 0$, as $n \to \infty$.

7) Let E be a topological vector space. Prove that if A and B are compact subsets of E, then so is $A + B$.

8) Consider $X \in \mathcal{D}(\mathbb{R}^2)^*$ defined by

$$\mathcal{D}(\mathbb{R}^2) \ni \phi \mapsto X(\phi) \equiv \int_{\mathbb{R}} \phi(x, -x)\,dx\,.$$

(a) Show that X is a distribution of order zero.
(b) What is the support of X? (Prove it) Use this to prove that X is not a regular distribution with continuous coefficient function.
(c) Show that $(\partial_1 - \partial_2)X = 0$.

Note. *So X is a solution of the PDE $(\partial_1 - \partial_2)X = 0$ which is not a classical solution.*