PH.D. QUALIFYING EXAM IN
MEASURE AND INTEGRATION THEORY
MAY 10, 2005

Give complete proofs and computations. Be particularly careful to indicate why the most crucial steps in your proof are true. Partial credit will be given where justified.

1) Let μ be a positive Radon measure on a locally compact Hausdorff space T. Let $f_n : T \to \overline{\mathrm{IR}}$ be μ-measurable for all $n \in \mathrm{IN}$. Prove that $\sup\{f_n \mid n \in \mathrm{IN}\}$ is μ-measurable.

2) Let E be a complex Banach space. Suppose $\{x_n\}_{n\in\mathrm{IN}} \subset E$ converges in the weakened topology $\sigma(E, E')$ to $x \in E$, *i.e.* $| < x_n - x, y > | \to 0$ for all $y \in E'$ as $n \to \infty$. Prove that x is contained in the closure of the linear span of the set $\{x_n\}_{n\in\mathrm{IN}}$.

3) Let E be a topological vector space. Prove the following:
(a) If $A \subset E$ is arbitrary and $U \subset E$ is open, then $A + U$ is open.
(b) If $A \subset E$ is compact and $B \subset E$ is closed, then $A + B$ is closed.
(c) The sum of closed subsets may fail to be closed.

4) Let $\delta \in \mathcal{D}'(\mathrm{IR})$ denote the distribution given by $\delta(\phi) = \phi(0)$, for all $\phi \in \mathcal{D}(\mathrm{IR})$.
(a) For which $f \in C^\infty(\mathrm{IR})$ is $f \cdot \delta' = 0$?
(b) What is the support of δ?
(c) If $X \in \mathcal{D}'(\mathrm{IR})$, $f \in C^\infty(\mathrm{IR})$ and f restricted to the support of X is 0, is it true that $f \cdot X = 0$?
Prove your answers are correct.

5) Let $K : [0,1] \times [0,1] \to \mathrm{IR}$ be continuous and $f : [0,1] \times \mathrm{IR} \to \mathrm{IR}$ be continuous and bounded. Show that the Hammerstein equation

$$u(s) = \int_0^1 K(s,t) f(t, u(t))\, dt\,,\ s \in [0,1]\,,$$

has a solution in $E = \{x : [0,1] \to \mathrm{IR} \mid x \,\text{continuous}\}$.

6) Let $1 \in \mathcal{D}'(\mathrm{IR})$ be given by $1(\phi) = \int_{\mathrm{IR}} \phi\, dx$, for all $\phi \in \mathcal{D}(\mathrm{IR})$. Compute $(1 * \delta')(\phi)$ for all $\phi \in \mathcal{D}(\mathrm{IR})$.

7) Let μ be a σ-finite positive Radon measure on a locally compact Hausdorff space T. Let $\{f_n\}_{n\in\mathrm{IN}} \subset \mathcal{L}^p(T, \mu)$ for some $p \in [1, \infty)$. Suppose that $\lim_{n\to\infty} f_n(t) = f(t)$ μ-a.e., that $f \in \mathcal{L}^p(T, \mu)$ and that $\lim_{n\to\infty} \int |f_n|^p\, d\mu = \int |f|^p\, d\mu$. Prove that $\lim_{n\to\infty} \int |f_n - f|^p\, d\mu = 0$. Hint: Bring the Lebesgue Dominated Convergence Theorem and the σ-finiteness of the measure into play.

8) Let $\mathcal{H}$ be a Hilbert space with inner product $(\cdot, \cdot)$, and let $\{y_\alpha\}_{\alpha\in A} \subset \mathcal{H}$ be an orthonormal family. Let $x \in \mathcal{H}$.
(a) Prove that $\sum_\alpha (x, y_\alpha)\, y_\alpha$ converges strongly to an element $y \in \mathcal{H}$.
(b) Prove that the vector $x - y$ is orthogonal to the closure of the linear span of the set $\{y_\alpha\}_{\alpha\in A}$.