Functional Analysis Qualifying Exam, May 2003

Do All Nine

1. Suppose X, Y are topological vector spaces, $\Lambda : X \to Y$ is linear, and N is a closed subspace of X. Let $\pi : X \to X/N$ denote the quotient map. Show, if $\Lambda(x) = 0$ for each $x \in N$, then there is a unique $f : X/N \to Y$ satisfying $\Lambda = f \circ \pi$. Show further, f is linear and Λ is continuous if and only if f is continuous.

2. Show, if X is an infinite dimensional F-space (a topological vector whose topology is induced by a complete invariant metric), then X does not have a countable Hamel basis. A Hamel basis is a basis in the sense of linear algebra.

3. Is every weakly convergent sequence in ℓ^1 stongly convergent? Are the weak and strong topologies on ℓ^1 the same?

4. Let δ denote the distribution defined by $\delta(\phi) = \phi(0)$ for $\phi \in \mathcal{D}(\mathbb{R})$. Find the support of δ'. For which $f \in C^\infty(\mathbb{R})$ is $f\delta' = 0$. Is it possible that an $f \in C^\infty(\mathbb{R})$ can vanish on the support of an $\Lambda \in \mathcal{D}'(\mathbb{R})$ and yet $f\Lambda \neq 0$?

5. Construct a sequence in $\mathcal{D}(\mathbb{R})$ which converges to 0 in the topology of $\mathcal{S}_1$ (tempered distributions), but not in the topology of $\mathcal{D}(\mathbb{R})$.

6. Prove, if $\mathcal{A}$ is a finite dimensional Banach algebra with unit e, if $x, y \in \mathcal{A}$, and if $xy = e$, then $yx = e$.

7. Show, if X is a compact Hausdorff space, then there is a natural one-one correspondence between closed subsets of X and ideals in $C(X)$.

8. Let $\mathbb{T} = \{\gamma \in \mathbb{C} : |\gamma| = 1\}$ denote the unit circle. Let U denote the bilateral shift, viewed as the operator $Uf(\gamma) = \gamma f(\gamma)$ on $L^2(\mathbb{T})$ (with repsect to arclength measure). Given ω a measurable subset of $\mathbb{T}$, let P_ω denote the projection onto the subspace of functions which are supported in ω. Does P_ω commute with U? Find the spectral decomposition of U.

9. Let $\mathcal{A}$ denote the commutative Banach algebra of functions of the form

$$f(x) = \sum_{n \in \mathbb{Z}} f_n \exp(inx), \text{ with } \sum |f_n| < \infty$$

under pointwise multiplication and with the norm $\|f\| = \sum |f_n|$. Show, if, for each x, $f(x) \neq 0$, then f is invertible in $\mathcal{A}$. Is $\mathcal{A}$ a C^*-algebra under pointwise complex conjugation?