Ph.D. Measure and Integration Exam, January 03

Part I State the following theorems.

1 Hölder's inequality.

2 The Hahn Decomposition Theorem.

3 The Fubini Theorem for positive measures μ and ν.

4 The Monotone Class Theorem.

5 The Hahn-Banach Theorem.

6 The Monotone Convergence Theorem.

7 The completeness of $L^1_E(\mu)$.

8 The theorem of uniform boundedness.

Part II Prove one of either Theorem 3 or 7 from part I.

Part III Solve the following problems. In problems one through four (X, Σ, μ) is a measure space and F is a Banach space.

1 Let $\mathcal{F} \subset \Sigma$ be a σ-subalgebra. Prove the existence of the conditional expectation $E(f|\mathcal{F})$ for $f \in L^1_F(\mu)$.

2 Let $\mathcal{R}$ be a ring generating Σ and suppose $f \in L^1_F(\mu)$. Prove, if $\int_A f d\mu = 0$ for each $A \in \mathcal{R}$, then $f = 0$, μ almost everywhere.

3 Suppose E (as well as F) is a Banach space and U is a bounded linear operator from E to F. Show, if $f \in L^1_E(\mu)$, then $U \circ f \in L^1_F(\mu)$ and

$$\int U \circ f d\mu = U \int f d\mu.$$

4 Suppose X is a locally compact Hausdorff space. Show, if K is a compact set and the restriction of f to K, $f|_K$, is continuous, then $f\phi_K$ is a Borel function. Here ϕ_K is the characteristic function of K.

5 Prove that $C([0,1])$ is a closed proper subspace of $L^\infty([0,1])$ (the L^∞ space with respect to Lebesgue measure on the interval $[0,1]$). Explain how to view $L^1([0,1])$ as a subspace of $L^\infty([0,1])^*$. Is $L^1([0,1])$ a proper subspace of $L^\infty([0,1])^*$?