PhD Exam. in Analysis Fall 2002

(I) State the following theorems

1. The Egorov th.
2. The Lebesgue th.
3. The dual of $L^p(\mu)$, $1 \le p < \infty$.
4. The Fubini theorem for $\mu\times\nu$- integrable functions
5. The Riesz representation theorem (of linear cont. functional on the space $C(K)$ of cont. functions on a compact space K).

(II) State and prove one of the following 2 theorems

1. The representation of the dual of $L^1(\mu)$.
2. The Fubini theorem for $\mathcal{S}\times\mathcal{T}$- measurable functions

(III) Solve the following problems.

Let (X, Σ, μ) be a measure space and E a Banach space

1. Assume μ is finite. Let $\mathcal{F} \subset \Sigma$ be a sub σ- algebra. Prove the existence of the conditional expectation $E(f|\mathcal{F})$ for functions $f \in L^1_E(\mu)$.
2. Let $\mathcal{R} \subset \Sigma$ be an algebra generating Σ and $f \in L^1_E(\mu)$. Prove that there is a sequence (f_n) of $\mathcal{R}$- step function converging to f, μ-a.e. and in the mean
3. Let $\mathcal{R} \subset \Sigma$ be an algebra ~~[illegible]~~ generating Σ and $f \in L^1_E(\mu)$. Prove that if $\int_A f\,d\mu = 0$ for every $A \in \mathcal{R}$, then $f = 0$, μ-a.e.
4. Assume μ is finite and let $f_n, f \in L^1_E(\mu)$, $n = 1, 2, \dots$ Prove that if $f_n \to f$ uniformly, then $f \in L^1_E(\mu)$ and $f_n \to f$ in the mean.
5. Let $(X, \mathcal{S})$, $(Y, \mathcal{T})$ be measurable spaces, $\mu: \mathcal{S} \to \mathbb{R}$ and $\nu: \mathcal{T} \to \mathbb{R}$ σ- additive measures. Prove the existence of the product measure $\mu\times\nu$ and the equality

$|\mu\times\nu| = |\mu| \times |\nu|$

Note. Write complete computations and explanations
Do not use arrows or other unnecessary signs.
Use words for explanations.
Write complete sentences.
Write complete statements of theorems (including
the framework).