PhD EXAM IN ANALYSIS . May 2002.

(I) State the following theorems:

1. The Egorov theorem
2. The Radon-Nikodym theorem
3. Density of step functions in L^p and L^∞
4. The Fubini theorem for $\mu\times\nu$-integrable functions
5. The Riesz representation theorem of the dual of $C(K)$.

(II) State and prove one of the following theorems

1. The integral representation of the dual of $L^1(\mu)$ (the finite measure case).
2. The theorem of integration with respect to $g\mu$.

(III) Solve the following problems:

1. Let (X, Σ, μ) be a measure space with $\mu(X) < \infty$, F a Banach space and $\mathcal{F} \subset \Sigma$ a σ-algebra. Prove the existence of the conditional expectation $E(f|\mathcal{F})$ for $f \in L^1_F(\mu)$.
2. Let $(X, \mathcal{S}), (Y, \mathcal{T})$ be measurable spaces and $\mu: \mathcal{S} \to \mathbb{R}$, $\nu: \mathcal{T} \to \mathbb{R}$ σ-additive measures. Prove the existence of the product measure $\mu \times \nu$ and the equality $|\mu\times\nu| = |\mu| \times |\nu|$.
3. Let $(X, \mathcal{S}), (Y, \mathcal{T})$ be measurable spaces, $p: X \to Y$ an $(\mathcal{S}, \mathcal{T})$ measurable mapping, $\mu: \mathcal{S} \to \mathbb{R}_+$ a σ-additive measure and $\nu = p(\mu)$ the image measure of μ under the mapping p. Prove that if $f \in L^1_F(\nu)$, then $f \circ p \in L^1_F(\mu)$ and $\int f\circ p \, d\mu = \int f \, dp(\mu)$.

4. Let (X, Σ, μ) be a measure space and $f \in L^1(\mu)$. Prove that: for every $\varepsilon > 0$, there is a set $X_\varepsilon \in \Sigma$ with $\mu(X_\varepsilon)$, such that
$$\int_{X - X_\varepsilon} |f| \, d\mu < \varepsilon.$$

5. Let λ be the Lebesgue measure on $[0,1]$ and $f \in L^1(\lambda)$. Prove that if
$$\int_{[0,1]} x^n f \, d\lambda = 0 \quad \text{for every } n, \text{ then } f = 0, \lambda\text{-a.e.}$$
Hint: Use the fact that the continuous functions on $[0,1]$ are dense in $L^1(\lambda)$ and reduce to the case $\int_A f \, d\mu = 0$ for $A \in \mathcal{B}([0,1])$.

Note. Write complete computations and explanations.
Do not use arrows or other unnecessary signs. Use words for explanations.
Write complete sentences.
State the complete framework for each theorem