PhD Exam in Analysis, Sept. 2001

(I) State the following theorems:

1. The monotone class theorem
2. The Hahn Decomposition theorem
3. The Egorov Theorem
4. The Vitali Convergence theorem
5. The Lebesgue convergence theorem
6. The Radon Nikodym theorem
7. The Fubini theorem
8. The dual of L^p
9. The Hahn-Banach theorem
10. Integrability with respect to the measure $g\mu$ defined by $g\mu(A) = \int_A g\,d\mu$.

(II) Prove 2 of the above theorems 3, 5, 6, 7.

(III) Solve the following problems. Let (X, Σ, μ) be a measure space.

1. Let X, Y be normed spaces. Prove that if Y is complete, then $L(X,Y)$ is complete.
2. Prove that if $f: X \to \overline{\mathbb{R}}_+$ is μ-measurable and $\int f\,d\mu < \infty$, then $f < \infty$, μ-a.e.
3. If $\mu(X) < \infty$, $f_n \in L^1(\mu)$ and $f_n \to f$ uniformly, then $f \in L^1(\mu)$ and $f_n \to f$ in the mean
4. Prove the existence of the conditional expectation $E(f \mid \mathcal{F})$ for functions $f \in L^1(\mu)$, with respect to a σ-algebra $\mathcal{F} \subset \Sigma$.
5. Prove that if $f \in L^1$ and $\int_A f\,d\mu = 0$ for A in a ring R generating the σ-algebra Σ, then $g = 0$, μ-a.e.
6. Prove the existence of the product $\mu \times \nu$ of two real valued measures.

<u>Note</u>. Write complete computations and explanations. Do not use arrows or other unnecessary signs. Use words for explanations. Write complete sentences.