May 2001

PhD Exam in Analysis

(I) State the following theorems:

1. the Hahn-Banach theorem
2. The monotone class theorem
3. The monotone convergence theorem in $L^1(\mu)$.
4. The Lebesgue dominated convergence theorem in $L^p(\mu)$.
5. The density of step functions in $L^p(\mu)$, $1 \le p < \infty$ and $L^\infty(\mu)$
6. The computation of the seminorm $\|f\|_p$
7. The integrability with respect to a measure defined by density.
8. the Fubini theorem for $\mu \times \nu$- integrable functions.
9. The Lusin theorem
10. The Riesz representation theorem.

(II) State and prove the following theorems:

1. The Egorov theorem
2. The integral representation of the dual of $L^1(\mu)$, μ finite.

(III) Solve the following problems:

1. Prove that if X is a normed space and Y is a Banach space, then $L(X,Y)$ is a Banach space.
2. Prove the existence of the conditional expectation for functions $f \in L^1(\mu)$ and for functions $f \in L^1_F(\mu)$.
3. Prove that the set of step functions is dense in $L^\infty(\mu)$.
4. Let (X, Σ, μ) be a measure space and $\mathcal{R}$ a ring generating the σ-algebra Σ. Let $g : X \to \mathbb{R}$ be a μ-measurable function. Prove that if $\int_A g\,d\mu = 0$ for every $A \in \mathcal{R}$, then $g = 0$, μ-a.e. Is this true for $g : X \to F$, where F is a Banach space?
5. Let (X, Σ) be a measurable space and (μ_n) a sequence of real valued measures on Σ. Define $\lambda : \Sigma \to \mathbb{R}_+$ by

$$\lambda(A) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|\mu_n|(A)}{1 + |\mu_n|(X)}, \text{ for } A \in \Sigma$$

Prove that λ is finite, σ-additive on Σ and $\mu_n \ll \lambda$ for each n.

Note. Write complete computations and explanations.
Do not use arrows or other unnecessary signs.
Use words for explanations.
Write complete sentences.
• State the complete framework for each theorem and problem,