Ph. D. Exam in Measure & Integration Theory Sept 2000.

Print your name on each sheet turned in. Write all proofs in a neat and logical fashion.

1. State and prove the Radon-Nikodym Theorem.

2. State and prove the Vitali Integral Convergence Theorem.

3. Let (X, Σ, μ) be a finite measure space and suppose $\Sigma = \sigma(\Sigma_0)$, where Σ_0 is an algebra. Show that for every $\varepsilon > 0$, if $E \in \Sigma$, then there exists an $E_0 \in \Sigma_0$ such that $\mu(E \Delta E_0) < \varepsilon$

4. Let Y and Z be closed subspaces in the B-space X. Suppose each $x \in X$ has a unique representation in the form $x = y + z$, with $y \in Y$ and $z \in Z$. Show that there exists a constant K such that $|y| \leq K|x|$ and $|z| \leq K|x|$, for each $x \in X$. [Hint: First apply Closed Graph Thm to $T: X \to Y$, $T(x) = y$].

5. Let (X, Σ, μ) be a measure space. Let (f_n) be a sequence of integrable functions such that $\sum \int |f_n| d\mu < \infty$. Does $\sum f_n$ converge a.e.? Prove.

6. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let X and Y be integrable random variables. Suppose X and Y are independent. Find $E(X/Y)$.

7. Let (X, Σ, μ) be a finite measure space; let (f_n) be a sequence of integrable functions such that $\int_E f_n d\mu \to$ for each $E \in \Sigma$.
Assume (f_n) is L^1 bounded. Show there exists an integrable function f such that $\int_E f_n d\mu \to \int_E f d\mu$ for all $E \in \Sigma$.