# Ph D Exam in Analysis May, 2000

Be sure to write all proofs in a neat and logical fashion. Give reasons for all steps!

① State and prove the Vitali Convergence Theorem for integrals.

② State and prove the Martingale Convergence theorem.

③ Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ be finite measure spaces. Construct $\mu \otimes \nu$ on $\mathcal{S} \otimes \mathcal{T}$. Give details.

④ Let $\{\mu_n\}$ be a sequence of real valued signed measures on $(S, \Sigma)$. Define $\lambda(A) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|\mu_n|(A)}{1+|\mu_n|(S)}$, $A \in \Sigma$. show $\lambda$ is a finite measure on $\Sigma$ and $\mu_n \overset{\varepsilon\text{-}\delta}{\ll} \lambda$ for each $n$.

⑤ Let $f : [0, b] \to \mathbb{R}$ be Lebesgue integrable. Suppose $\int_{[0,c]} f \, dm = 0$ for each $c \in [0, b]$. Show $f = 0$ a.e. $m$.

⑥ Let $(\Omega, \mathcal{F}, P)$ be a pr. space, $Y$ an integrable r.v. and $\mathcal{E}$ a sub $\sigma$-field of $\mathcal{F}$. Suppose $\mathcal{E}$ and $\mathcal{F}(Y)$ are independent. What is $E(Y \mid \mathcal{E})$? Prove.

⑦ Let $\mathfrak{X}$ be a B-space and suppose $\mathfrak{Y}$ and $\mathfrak{Z}$ are closed linear subspaces of $\mathfrak{X}$ such that each $x \in \mathfrak{X}$ has a unique representation of the form $x = y + z$, where $y \in \mathfrak{Y}$ and $z \in \mathfrak{Z}$. Show that there is a constant $K$ such that $|y| \le K|x|$ and $|z| \le K|x|$, for each $x \in \mathfrak{X}$ where $y$ & $z$ are given as above.

[Hint: define a map $T : \mathfrak{X} \to \mathfrak{Y}$ which is useful, use the closed graph theorem to get continuity; then define $S : \mathfrak{X} \to \mathfrak{Z}$, etc.]