PH.D. EXAM in ANALYSIS

May 11, 1999.

(I) State the following theorems:

1.) Hahn-Banach
2.) The Egorov theorem
3.) The Vitali convergence theorem
4.) The Radon-Nikodym theorem
5.) The Fubini theorem
6.) The monotone convergence theorem
7.) The completeness theorem for $\mathcal{L}^1_F(\mu)$.
8.) The equality $|gm| = |g||m|$.

(II) Prove one of the theorems 3 or 7.

(III) Solve the following problems, where (X, Σ, μ) is a measure space and F is a Banach space.

1.) If $f_n \to f$ in $\mathcal{L}^1_F(\mu)$ and $f_n \to g$, μ-a.e., then $f = g$, μ-a.e.

2.) Let $\mathcal{F} \subset \Sigma$ be a σ-algebra. Prove the existence of the conditional expectation $E(f/\mathcal{F})$ for $f \in \mathcal{L}^1_F(\mu)$

3.) Let $\mathcal{R}$ be a ring generating Σ and $f \in \mathcal{L}^1_F(\mu)$. Prove that if $\int_A f d\mu = 0$ for every $A \in \mathcal{R}$, then $f = 0$, μ-a.

4.) Let $(X, \mathcal{F})$ and $(Y, \mathcal{T})$ be measurable spaces, $\mu : \mathcal{F} \to \mathbb{R}$ and $\nu : \mathcal{T} \to \mathbb{R}$, σ-additive measure. Prove the existence on the product measure $\mu \times \nu : \mathcal{F} \times \mathcal{T} \to \mathbb{R}$ satisfying $\mu \times \nu (A \times B) = \mu(A)\nu(B)$ for $A \times B \in \mathcal{F} \times \mathcal{T}$ and $|\mu \times \nu| = |\mu| \times |\nu|$.

5.) If $f \in \mathcal{L}^1_F(\mu)$ there is a sequence (f_n) of $\mathcal{R}$-step functions

Note. Write complete computations and explanation
Do not use arrows or other unnecessary signs. Use
Words for explanations.
Write complete sentences.