PhD Exam in Analysis

May 1997

Be sure to write all steps in a neat and logical fashion to receive credit. Give reasons for your steps.

<u>Definition</u> A σ-algebra $\mathcal{F}$ is <u>separable</u> if $\mathcal{F} = \sigma\{A_j : j \geq 1\}$ for some sequence of sets A_j.

① Let $\{\mathcal{F}_k\}$ be a sequence of separable σ-algebras of subsets of a set X. Let $\mathcal{F}$ be the smallest σ-algebra of subsets of X which contains $\mathcal{F}_k$ for each k. Prove that $\mathcal{F}$ is separable.
[Hint: Use the Monotone Class Theorem].

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② Let $(X, \mathcal{F}, \mu)$ be a finite measure space and let (Y, Σ) be a measurable space. Let $f : X \to Y$ be $\mathcal{F}, \Sigma$-measurable. Define $\nu(B) = \mu(f^{-1}(B))$ for $B \in \Sigma$. Show ν is a measure on Σ and if $g : Y \to [0, \infty)$ is Σ-measurable, then $\int_Y g \, d\nu = \int_X g \circ f \, d\mu$

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③ Let m be Lebesgue measure on $\mathbb{R}$. Let $f_n = -1_{A_n}$, where $A_n = (n, \infty)$. Show (f_n) is increasing, but the monotone convergence theorem is not true. Why is this?

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④ Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{J}, \nu)$ be two finite measure spaces. Construct the product measure $\mu \times \nu$ on $\mathcal{S} \otimes \mathcal{J}$. Be sure to prove $\mu \times \nu$ is countably additive.

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⑤ Let (X, Σ, μ) be a finite measure space. Prove that the dual of $L^1(X, \Sigma, \mu)$ is isometrically isomorphic to $L^\infty(X, \Sigma, \mu)$

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⑥ Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Suppose that $\mathcal{F}$ is separable. Show $L^1(X, \mathcal{F}, \mu)$ is separable.

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⑦ Let $(X, \mathcal{F}, \mu)$ be a measure space. Let $\{f_n\}$ be a sequence of measurable functions such that $\int_X |f_n| \, d\mu < \frac{1}{2^n}$. Does it follow that $\sum_1^\infty f_n$ converges a.e. μ on X?

(8) Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $X : \Omega \to \mathbb{R}$ be an integrable function. Let $F(x) = P(X < x)$, for $x \in \mathbb{R}$.

(a) Show $F : \mathbb{R} \to [0,1]$ is monotone, left continuous, $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$.

(b) Show $\int_\Omega X^2 dP = \int_{\mathbb{R}} x^2 dF(x)$

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(9) Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let (X_n) be a sequence of independent random variables on Ω.

(a) Show $(\overline{\lim} X_n > 3)$ is a tail event relative to (X_n).

(b) If $P(\overline{\lim_n} X_n > 3) > 0.2$, what is $P(\overline{\lim_n} X_n > 3)$?

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(10) Let ℓ^∞ be the Banach space consisting of all real sequences (a_i) such that $\sup_i |a_i| < \infty := \|(a_i)\|$.
Let c be the subspace of ℓ^∞ consisting of (a_i) such that $\lim a_i$ exists.

Prove that there exists a continuous linear functional $F : \ell^\infty \to \mathbb{R}$ such that $F((a_i)) = \lim a_i$, for each $(a_i) \in c$.

[Hint: Use the Hahn-Banach theorem].