Ph.D. Examination
in
Measure Theory and Integration

June 7, 1993

Remark. In the sequel the word *measure* means positive, countably additive measure, unless otherwise stated.

Instructions. **Do all of the following problems.**

Problem 1. Let $f_n : \mathbf{R} \to \mathbf{R}$ be a sequence of Lebesgue-measurable functions which converges pointwise to a function g. Prove, directly from the definition of measurability, that g is measurable.

Problem 2. For $f \in L^1(\mathbb{R})$, let f_t denote the translation of f by t, i.e. $f_t(x) := f(x-t)$. Show that the map $t \mapsto f_t$ is a continuous map from $\mathbb{R}$ to $L^1(\mathbb{R})$.

Problem 3. Let μ denote the Lebesgue measure on $\mathbf{R}$. Let $T : \mathbf{R} \to \mathbf{R}$ be defined by $T(x) = x^3 - x$. Define a Borel measure ν setting $\nu(A) := \mu(T^{-1}A)$. Compute the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$.

Problem 4. Let $K_n \subset [0,1]$ be a Cantor set of Lebesgue measure larger than $1 - \frac{1}{n}$. Let $V := \bigcup_{n=1}^{\infty} K_n$. Prove that, for any nullset N, the set $V \setminus N$ is not a G_δ set. [Hint: Baire Category Theorem]

Problem 5. Let μ and ν be σ-finite measures on the measurable space (X, B). Prove, from first principles, that ν can be written as $\nu = \nu_{ac} + \nu_{sing}$, with $\nu_{ac} \ll \mu$ and $\nu_{sing} \perp \mu$.

Problem 6. Let x_n be a sequence in a Banach space B. Assume that that the sequence x_n converges weakly to x. (a) Show that x is the only weak limit of the sequence x_n. (b) Show that $\sup_n \|x_n\| < \infty$. [Hint for (b): Consider the x_n as elements of the double dual X^{**}]

Problem 7. Let $K : [0, 2\pi] \to \mathbf{R}$ be defined by $K(x) = \frac{1}{4}\sin(x)$. Show that for every $h \in L^p([0, 2\pi])$, there exists a unique solution f to the equation

$$f + f * K = h.$$

If h is C^∞, is it true that the solution f is C^∞? Justify your answer.

Problem 8. A distribution T is called harmonic if $\Delta T = 0$, where $\Delta = \sum_1^n \frac{\partial^2}{\partial x_i^2}$. Show that if T is a harmonic tempered distribution, then T is a polynomial. [Hint: a distribution with support $\{0\}$ is a finite sum of multiples of Dirac's δ and its derivatives.]