

PHD EXAMINATION IN ANALYSIS

August 1995

All work must be presented in a neat and logical fashion to receive credit. Be sure to give reasons for all your steps. DO NOT leave any gaps! Put your name on each sheet turned in.

1. (X, Σ, μ) will always denote a measure space (possibly infinite). Prove that $L^1(S, \Sigma, \mu)$ is complete.
2. Let f be Lebesgue integrable on \mathbb{R} . Suppose $\int_0^x f dm = 0$ for all real x , where m is Lebesgue measure. What can you say about f ? Prove your answer.
3. Let (f_n) be a sequence of integrable functions such that $\sum_{n=1}^{\infty} \int |f| d\mu < \infty$. What can you prove about the a.e. μ convergence of $\sum f_n$?
4. Let f be a Lebesgue integrable function on \mathbb{R} . Let $\varepsilon > 0$. Is it possible to find a bounded interval I such that whenever E is a measurable set such that $E \cap I = \emptyset$, then $|\int_E f m| < \varepsilon$?
5. State and prove the Lebesgue Monotone Convergence Theorem.
6. Present the steps in the construction of the product measure of two finite measure spaces. Sketch the proof of the countable additivity of the product measure.
7. Let $X = Y =$ any uncountable set. Let $\mathcal{S} = \mathcal{T} =$ the class of all, countable subsets of X and Y respectively. Note that \mathcal{S} and \mathcal{T} are σ -rings. Let $D = \{(x, y) : x = y\}$ be the diagonal in $X \times Y$. Show that D is NOT $\mathcal{S} \times \mathcal{T}$ measurable.
8. Let (X, Σ, μ) be a finite measure space and \mathcal{G} is a sub σ -algebra of Σ . Define the closed subspace V of $L^\infty(X, \Sigma, \mu)$ as the collection of all $g \in L^\infty(X, \Sigma, \mu)$ such that $\int fg \mu = 0$ for all $f \in L^1(x, \mathcal{G}, \mu)$. Prove that if $k \in L^1(X, \Sigma, \mu)$ and $\int kg d\mu = 0$ for all $g \in V$, then k is \mathcal{G} -measurable. (Hint: Use the representation for the dual of $L^1(X, \Sigma, \mu)$ and a corollary of the Hahn-Banach theorem applied to the closed subspace $L^1(x, \mathcal{G}, \mu)$ of $L^1(X, \Sigma, \mu)$).