Ph.D. Exam in Integration Theory

October 24, 1994

$(X, \sum, \mu)$ is a finite measure space.

I. State the following theorems:
 1.) Lebesgue dominated convergence theorem
 2. Fatou Lemma
 3. Vitali theorem
 4. Monotone convergence theorem
 5. The Radon Nikodym theorem
 6. The Egorov theorem
 7. The Fubini theorem
 8. The Lebesgue decomposition theorem
 9. The Uniform boundedness theorem
 10. The Hahn Banach theorem

II. Prove the $(L^1)^* = L^\infty$

III. Problems

1. Find $\lim_{n\to\infty} \int_0^1 nx^n dx$.
 (Hint: use the monotone convergence theorem)
2. Let μ be the Lebesgue measure on $\mathbf{R}$ and f a Lebesgue–integrable function on $\mathbf{R}$, such that $\int_0^x f d\mu = 0$ for every $x \in \mathbf{R}$. What can you say about f?
3. Let $f \geq 0$ be μ–integrable such that $\int f d\mu \leq 1$. Prove the $\int \frac{1}{f} d\mu \geq 1$. (Hint, use the Schwartz inequality for $\sqrt{f}$ and $\frac{1}{\sqrt{f}}$).
4. Let R be a ring of subsets of X, E a Banach space and $\mu : R \to E$ be a finitely additive measure. Prove that μ is σ–additive iff for any decreasing sequence (A_n) from R with $\bigcap_n A_n = \phi$ we have $\lim \mu(A_n) = 0$.