Ph.D. Exam on Integration Theory

May 9, 1994

I. State the following theorems:

1) The Uniform Boundeness theorem
2) The Hahn–Banach theorem
3) The Faton Lemma.
4) The Lebesgue dominated convergence theorem
5) The Vitali theorem
6) The Randon–Nikodym theorem
7) The Egorov theorem
8) The Fubini theorem
9) The dual of $L^P (1 \leq p < \infty)$
10) The Lusin theorem
11) The Riesz representation theorem in locally compact spaces
12) The Lebesgue decomposition theorem

II. Prove one of the theorems 8 or 9.

III. Solve the following problems:
(X, Σ, μ) is a measure space, E is a Banach space.

1) Let (f_n) be a sequence of $L^1_E(\mu)$ such that $\int |f_n| d_\mu < \frac{1}{n^2}$ for every n. Prove or disprove the pointwise convergence and the convergence in the mean of the series Σf_n.

2) Assume μ is finite and let R be a ring generating Σ. Define $\rho(A, B) = \mu(A \Delta B)$ for $A, B \in \Sigma$. Prove that ρ is a semi-distance and that R is dense in Σ for ρ.

3) Let $X = \mathbb{R}$ and μ the Lebesgue measure. Let $f \in \mathcal{L}^1_E(\mu)$ such that $\int_0^x f d\mu = 0$ for every $x \in \mathbb{R}$ (if $x < 0$, $\int_0^x = -\int_x^0$). Prove that $f = 0, \mu$–a.e.

4) Let μ, ν be finite, σ–additive measures on Σ, such that $\mu \ll \nu$ and $\nu \ll \mu$. What can be said about the relationship of the Radon–Nikodym densities of one measure with respect to the other.

5) Let (x_n) be a sequence in the Banach space E. Assume $x_n \to x$ weakly in E. Show that x is the only weak limit of the sequence (x_n) and that $\sup_n \|x_n\| < \infty$. (Hint: Consider the x_n as elements of the bydual E^{**}).

6) Let $X = \mathbb{R}$ and μ the Lebesgue measure. Let $f \in L^1(\mu)$ and for each $t \in \mathbb{R}$ define $f_t(x) = f(x - t)$. Show that the mapping $t \mapsto f_t$ is a continuous map from $\mathbb{R}$ into $L^1(\mu)$. (Hint: prove it first for continuous functions with compact support).