Ph.D. Exam in Measure Theory
January, 1993

Be sure to present all work in a neat and logical fashion in order to receive credit.

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1. State and prove the Radon-Nikodym theorem.

2. Let (S, Σ, μ) be a positive finite measure space. Prove that $\left(L'(S, \Sigma, \mu)\right)^* = L^\infty(S, \Sigma, \mu)$

3. Give the construction of the product measure $\mu \times \nu$ on $\mathcal{S} \otimes \mathcal{W}$, where $(\mathcal{X}, \mathcal{S}, \mu)$ and $(\mathcal{Y}, \mathcal{W}, \nu)$ are finite non negative measure spaces. (Be sure to prove that $\mu \times \nu$ is countably additive).

4. Let (S, Σ, μ) be a positive measure space with $\mu(S) = 1$. Let $\mathcal{A}$ and $\mathcal{B}$ be a sub σ-algebras of Σ. Suppose $\mu(A \cap B) = \mu(a)\mu(B)$ for every $A \epsilon \mathcal{A}$ and $B \epsilon \mathcal{B}$. Show that $\int fg d\mu (\int f d\mu)(\int g d\mu)$, for $f \geq 0, g \geq 0$, where f is A-measurable and g is $\mathcal{B}$-measurable.

5. Let (S, Σ) be a measurable space, and suppose μ and ν are positive finite measures on Σ. Suppose $\mu \ll \nu$ and $\nu \ll \mu$.
   Show that $\dfrac{d\mu}{d\nu} = \dfrac{1}{\frac{d\nu}{d\mu}}$ a.e.ν.

6. Let (S_∞, Σ, μ) be a finite measure space. Let $(f_n)_{n=1}$ be a sequence of integrable function such that $\sum\limits_{n=1}^{\infty} \int_S |f_n| d\mu < \infty$.
   Does it follow that Σf_n converges pointwise on S a.e. μ? Show all details.

7. Let f be a Lebesque integrable function on $\mathbf{R}$. Let $\epsilon > 0$. Is it possible to find a bounded interval I such that whenever E is a measurable set such that $E \cap I = \emptyset$, then

$$|\int_E f dm| < \varepsilon?$$

   Show all work.

8. Let f be Lebesque integrable on [0, 1]. Suppose that $\int_0^x f dm = 0$, for every $x \epsilon [0, 1]$. What can you conclude about f? Prove your answer?