Ph.D. Exam in Analysis
Sept 1991

Present all work in a neat and logical fashion. Put each problem on a separate page. Print name on each sheet.

1. State and prove the Lebesgue Monotone Convergence Theorem.

2. Prove that $L^1(S, \Sigma, \mu)$ is complete, where μ is a nonnegative measure.

3. Give the construction of the product measure $\mu \times \nu$ on $\mathcal{S} \times \mathcal{W}$, where $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{W}, \nu)$ are $\sigma-$finite nonnegative measure spaces.

4. Let f be Lebesgue integrable on $\mathbb{R}$. Suppose $\int_0^x f dm = 0$ for all real x. What can you say about f? Prove your answer.

5. Let f be a Lebesgue integrable function on $\mathbb{R}$. Let $\varepsilon > 0$. Is it possible to find an interval I such that whenever E is a measurable set such that $E \cap I = \phi$, then $|\int_E f dm| < \varepsilon$?

6. Let X be a metric space. Let $\{P_n\}$ be a sequence of regular probability measures on $\mathcal{B}(X)$, the Borel subsets of X. Let $P = \sum_{n=1}^{\infty} \left(\frac{1}{2^n}\right) P_n$. Show that P is regular on $\mathcal{B}(X)$.

7. Let $\mu(dx) = x^2 dx$ on $[0, 1]$. Let $T : [0, 1] \to [0, 1]$ be defined by $T(x) = x^4$. Compute the Radon-Nikodym derivative of the image measure $T\mu$.

8. Let $(\mu, \mathcal{F}, \mu)$ be a measure space with $\mu(\Omega) = 1$.
Let $\mathcal{A}$ and $\mathcal{B}$ be sub $\sigma-$algebras of $\mathcal{F}$:
Suppose $\mu(A \cap B) = \mu(A)\mu(B)$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Prove that $\int fg d\mu = (\int f d\mu)(\int g d\mu)$ for all $\mathcal{A}$-measurable positive functions f and all $\mathcal{B}-$measurable positive functions g.