Ph. D. Examination in Measure and Integration.

January 23, 1989, 8:30 a.m. – 12:30 p.m..

Instructions: Answer 6 out of the 7 problems. Justify all work. To obtain any partial credit, all work must be presented in a neat, logical and concise fashion. Start each problem on a new sheet of paper.

Problem 1. Let f be a Borel–measurable function defined on [0,1] with $\infty > f(x) \geq 0$.

Part (a) Prove that there is a non–increasing function f^* defined on $[0,1]$ such that $m\{x : f^*(x) \geq \lambda\} = m\{x : f(x) \geq \lambda\}$ for every $\lambda \geq 0$, where m is Lebesgue measure on $[0,1]$. (f^* is called a monotone rearrangement of f.)

Part (b) Show that $\int f^* \, dm = \int f \, dm$.

Problem 2. Let $(X, \mathcal{F}, \mu)$ be a finite measure space, and let f and g be two measurable functions such that

$$\mu(f \in A, \ g \in B) = \mu(f \in A)\mu(g \in B)$$

for every $A, \ B \in \mathcal{B}(\mathbf{R})$. Let $\nu(A) = \mu(g \in A)$. Show that

$$\int H(f,g)d\mu = \int \int H(f,t)\nu(dt)d\mu$$

for every positive $\mathcal{B}(\mathbf{R}^2)$–measurable function H.

Problem 3. Consider $\ell^\infty = \{(a_n)_{n=1}^\infty : \ sup_n \ |a_n| < \infty\}$. This space becomes a Banach space when equipped with the norm $\| \ (a_n)_{n=1}^\infty \ \| = sup\{|a_n| : \ n \geq 1\}$.

Part (a) Show there is a linear functional $T : \ell^\infty \to \mathbf{R}$ with the property: if $lim_{n\to\infty} \, a_n$ exists, then $T[(a_n)_{n=1}^\infty] = lim_{n\to\infty} \, a_n$. (Use the Hahn-Banach theorem.)

Part (b) Modify your proof in (a) to show there is a linear functional $S : \ell^{\infty} \to \mathbb{R}$ with the property: if $lim_{n\to\infty} n^{-1} \sum_{k\le n} a_k$ exists, then $S[(a_n)_{n=1}^{\infty}] = lim_{n\to\infty} n^{-1} \sum_{k\le n} a_k$.

Problem 4. State and prove the Radon-Nikodym theorem.

Problem 5. State and prove Fubini's theorem.

Problem 6. Let μ be a measure on $((0,\infty), \mathcal{B}(0,\infty))$ defined by $\mu(A) = \int_A |x|^{-1/2}dx$. Let $T : (0,\infty) \to (0,\infty)$ be given by $T(x) = x^{-1}$. Define a new measure π on $(0,\infty)$ by setting $\pi(A) = \mu(T^{-1}(A))$. Compute $d\pi/d\mu$.

Problem 7. Let $c(t)$ be a function defined on $[0,\infty)$ with the following properties: $c(t)$ is strictly increasing and continuous, $c(0) = 0$, and $c(\infty) = \infty$. Let $\tau_t = inf\{s : c(s) > t\}$. Prove that

$$\int_0^{\infty} f(t)c(dt) = \int_0^{\infty} f(\tau_t)dt$$

for every positive continuous function f on $[0,\infty)$.