Ph. D. Examination in Measure and Integration.

May 9, 1988 6 - 10 p.m. Little 125

Instructions: Answer seven out of the eight questions. Justify all work. to obtain any partial credit, all work must be presented in a neat and logical fashion.

Question 1. Prove that $L^1(S, A, \mu)$ is a complete space.

Question 2. State and prove the Hahn-Banach theorem.

Question 3. State and prove Fubini's theorem.

Question 4. Prove that there exists no sequence $(c_n)_{n=1}^{\infty}$ of real numbers such that an infinite series $\sum a_n$ of real numbers converges if and only if $(c_n a_n)_{n=1}^{\infty}$ is a bounded sequence.

[Hint: Without loss of generality, assume such a sequence exists, with $c_n \neq 0$ for all n. Consider the mapping T: $\ell_\infty \to \ell_1$ given by $T((x_n)) = \frac{x_n}{c_n}$ and use the open mapping theorem.]

Question 5. Prove that there exists no Banach space of <u>algebraic</u> dimension $\aleph_0$

[Hint: show that a finite dimensional subspace is closed and then use the Baire category theorem.]

Question 6. Let f: R $\rightarrow$ R be a Lebesgue measurable function. Show that there exists a Borel measurable function g such that f = g a.e. (a.e. with respect to Lebesgue measure).

Question 7. Let μ be a measure on ($[0,\infty)$, $\mathcal{B}([0,\infty)$)) defined by

$$\mu(A) = \int_A x \; dx .$$

Let T: $[0,\infty) \rightarrow [0,\infty)$ be given by $T(x) = e^x - 1$. Define a new measure π on ($[0,\infty)$, $\mathcal{B}([0,\infty)$)) by setting $\pi(A) = \mu(T^{-1}(A))$. Compute $d\pi/d\mu$.

Question 8. Let λ be Lebesgue measure on (R, B(R)). If A $\in$ B(R), define the set $-A = \{ x : -x \in A \}$. Define $\mathcal{C} = \{ A \in \mathcal{B}(R) : A = -A \}$.

Part (a) Show $\mathcal{C}$ is a sigma-algebra.

Part (b) Define two new measures μ and ν on the measurable space (R, $\mathcal{C}$) by setting

$$\mu(A) = \int_A x \, dx \qquad \text{whenever } A \in \mathcal{C}.$$

$$\nu(A) = \int_{A \cap [0,\infty)} x \, dx \qquad \text{whenever } A \in \mathcal{C}.$$

Since λ may also be considered as a measure on $(\mathbb{R}, C)$, we can find two C-measurable functions $d\mu/d\lambda$ and $d\nu/d\lambda$ so that

$$\mu(A) = \int_A d\mu/d\lambda \; dx \qquad \text{whenever } A \in C$$

$$\nu(A) = \int_A d\nu/d\lambda \; dx \qquad \text{whenever } A \in C$$

<u>Compute these two functions.</u>