Ph.D. Examination in Measure and Integration
Jan 23, 1988

Answer all six questions. Justify all work. To obtain any partial credit, all work must be presented in a neat and logical fashion.

1. State and prove the Radon-Nikodym Theorem.

2. State and prove the Hahn-Banach Theorem.

3. Let Ω be a topological space, $\underline{B}(\Omega)$ the Borel subsets of Ω. Suppose that for each n, μ_n is a countably additive positive finite regular measure defined on $\underline{B}(\Omega)$. Let $\mu(\cdot) = \sum_{n=1}^{\infty} \frac{\mu_n(\cdot)}{2^n(1+\mu_n(\Omega))}$.
Show that μ is also a regular countably additive measure.

4. Let $\{f_n : n \geq 0\}$ and $\{g_n : n \geq -1\}$ be two sequences of real-valued measurable functions on a measurable space $(\Omega, \mathcal{F})$. Let $B \subset R^1$ be a Borel set on R^1. Define

$$\begin{aligned} T(\omega) &= \sup\{n \geq 0 : f_n(w) \in B\} \quad \text{if } \{n \geq 0 : f_n(w) \in B\} \text{ is bounded and nonempty} \\ &= 0 \qquad \text{if } \{n \geq 0 : f_n(w) \in B\} = \phi \\ &= -1 \qquad \text{if } \{n \geq 0 : f_n(w) \in B\} \text{ is unbounded.} \end{aligned}$$

(A) Prove T is $\mathcal{F}$-measurable.

(B) Define $h(w) = g_{T(w)}(w)$: prove h is $\mathcal{F}$-measurable.

5. Let $T : (-\pi/2, \pi/2) \to R$ be given by $T(x) = \tan x$.

Let λ be Lebesgue measure on $(-\pi/2, \pi/2)$ and define a measure μ on $(R, B(R))$ by setting $\mu(A) = \lambda(T^{-1}(A))$.

compute $\dfrac{d\mu}{dx}$.

6. Fix $1 \leq p < \infty$, and let $f \in L^p(X, \mathcal{A}, \mu)$. Show that

$$\|f\|_p = \sup\left\{ \int fg \, d\mu : \|g\|_q \leq 1 \right\}.$$

(Hint: 1. Show that if $\|g\|_q \leq 1$, then $\int fg \, d\mu \leq \|f\|_p$.

2. Show that $g = |f|^{p-1} \in L^q$.

3. State why doing 1. and 2. solves the problem.)