PhD Analysis Exam, January 2017

Do six of eight. Answer each problem on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

(1) Suppose $(X, \mathscr{M})$ and $(Y, \mathscr{N})$ are measurable spaces, $\mathscr{E} \subset \mathscr{N}$ and $f : X \to Y$. Show, if the σ-algebra generated by $\mathscr{E}$ contains $\mathscr{N}$ and $f^{-1}(E) \in \mathscr{M}$ for all $E \in \mathscr{E}$, then f is $\mathscr{M} - \mathscr{N}$ measurable.

(2) Suppose $(X, \mathscr{M}, \mu)$ is a measure space, $f : X \to [0, \infty)$ is measurable and $\int_X f < \infty$. For positive integers n, let $E_n = \{f \leq n\}$. Show,

(a) the sequence $(\int_{E_n} f)$ converges to $\int f$;

(b) for each $\epsilon > 0$ there is a positive integer M such that

$$\int_{E_M^c} f < \epsilon;$$

(c) for each $\epsilon > 0$ there is a $\delta > 0$ such that if $E \in \mathscr{M}$ and $\mu(E) < \delta$, then $\int_E f < \epsilon$.

(3) Let $(X, \mathscr{M}, \mu)$ be a measure space with completion $(X, \overline{\mathscr{M}}, \overline{\mu})$. Show, if $f : X \to \mathbb{R}$ is $\overline{\mu}$ measurable, then there exists a μ measurable $g : X \to \mathbb{R}$ such that $\overline{\mu}(\{x \in X : f(x) \neq g(x)\}) = 0$.

(4) Given $f \in L^1([0,1])$, let $g(x) = \int_x^1 \frac{f(t)}{t}\, dt$. Show $g \in L^1([0,1])$ and

$$\int_0^1 g(x)\, dx = \int_0^1 f(t)\, dt.$$

(5) Let X be a normed vector space (over $\mathbb{C}$). Suppose $\mathcal{M} \subset X$ is a closed subspace and $x \in X \setminus \mathcal{M}$. Prove the subspace $\mathcal{N} = \mathcal{M} + \mathbb{C}x$ is closed in X. Prove if $\mathcal{L} \subset X$ is a finite dimensional subspace, then $\mathcal{L}$ is closed in X.

(6) Suppose $(X, \mathscr{M}, \mu)$ is a σ-finite measure space. Prove the set of simple functions in $L^2(\mu)$ is dense in $L^2(\mu)$.

(7) State the Baire Category Theorem. Suppose X is a Banach space. Prove that, as a vector space, X does not have a countable basis. Show there is no norm $\|\cdot\|$ on c_{00} such that the normed vector space $(c_{00}, \|\cdot\|)$ is a Banach space.

(8) Suppose $(X, \mathscr{M}, \mu)$ is a σ-finite measure space. Recall that the mapping $\lambda : L^\infty(\mu) \to L^1(\mu)^*$ defined, for $g \in L^\infty(\mu)$, by $g \mapsto \lambda_g : L^1(\mu) \to \mathbb{C}$ where, for $f \in L^1(\mu)$,

$$\lambda_g(f) = \int_X fg \, d\mu,$$

is an isometric isomorphism. Prove the following special case of this duality result.

Suppose $(X, \mathscr{M}, \mu)$ is a finite measure space. Let $L^1_{\mathbb{R}}(\mu)$ denote the real Banach space of real-valued functions in $L^1(\mu)$. Suppose $\lambda \in L^1_{\mathbb{R}}(\mu)^*$ (the dual space) and $\lambda(f) \geq 0$ for each $f \in L^1_{\mathbb{R}}(\mu)$ such that $f \geq 0$ (pointwise). Prove there is an integrable function $g : X \to \mathbb{R}$ such that

$$\lambda(f) = \int_X fg \, d\mu.$$