Ph. D. Algebra Exam **May 3, 2016**

Answer seven problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Let M/K be a field extension and let p be a prime. Let L_1/K, L_2/K be finite subextensions of M/K such that $[L_1 : K]$ and $[L_2 : K]$ are both powers of p.
   (a) Prove that if L_1/K is a Galois extension then $[L_1L_2 : K]$ is a power of p.
   (b) Give an example which shows that if neither of L_1/K, L_2/K is Galois then $[L_1L_2 : K]$ need not be a power of p.
2. Prove that every field K is contained in an algebraically closed field.
3. (a) Let $\mathcal{C}$ be a category and let $f : X \to Y$ be a morphism in $\mathcal{C}$. Define what it means for f to have each of the following properties:
      i. f is a monomorphism.
      ii. f is an epimorphism.
      iii. f is an isomorphism.

   (b) Give an example of a category $\mathcal{C}$ and a morphism $f : X \to Y$ in $\mathcal{C}$ which is a monomorphism and an epimorphism, but not an isomorphism.
4. (a) Define what it means for a group G to be nilpotent.
   (b) Prove that if G is nilpotent and $N \trianglelefteq G$ then G/N is nilpotent.
   (c) Prove that if G is free on a set S and $|S| > 1$ then G is not nilpotent.
5. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence of groups (not necessarily abelian).
   (a) Suppose there is a group homomorphism $r : B \to A$ such that $r \circ f = \mathrm{id}_A$. Prove that $B \cong A \times C$.
   (b) Suppose there is a group homomorphism $s : C \to B$ such that $g \circ s = \mathrm{id}_C$. Prove that $B \cong A \rtimes_\phi C$ for some $\phi : C \to \mathrm{Aut}(A)$.

6. Let R be a commutative ring with 1 and let I be a proper ideal of R. The radical of I is defined to be

$$\sqrt{I} = \{x \in R : x^n \in I \text{ for some } n \geq 1\}.$$

Prove that $\sqrt{I}$ is equal to the intersection of all the prime ideals of R which contain I.

7. Let R be a commutative ring with 1 and let P and Q be projective R-modules. Prove that the R-module $P \otimes_R Q$ is projective.

8. Let R be an integral domain, let K be field of fractions of R, and let $A \subset K$ be a fractional ideal of R.

   (a) Prove that the set $A' = \{x \in K : xA \subset R\}$ is a fractional ideal of R.

   (b) Prove that the fractional ideal A is invertible if and only if $AA' = R$.

   (c) Give an example of an integral domain R and a nonzero fractional ideal A of R which is not invertible.

9. Let $D > 1$ be square-free. Determine the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{D})$. Your answer should be carefully justified.

10. Let R be a ring with 1 and let M be a unital R-module. Assume that for every R-submodule N of M there is an R-submodule N' such that $M = N \oplus N'$. Prove that if L is a nontrivial submodule of M then L contains a simple submodule.

11. Determine the number of isomorphism classes of *noncommutative* semisimple rings with $2592 = 2^5 \cdot 3^4$ elements. You do not have to list the rings.