Ph. D. Algebra Exam **May 6, 2015**

Answer seven problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Give an example of each of the following, with some explanation.
   (a) A field extension L/K which is algebraic but has infinite degree.
   (b) A field extension L/K which is normal but not Galois.
   (c) Two Galois field extensions L/K and M/L such that M/K is not Galois.

2. Let K be a splitting field over $\mathbb{Q}$ for the polynomial $f(X) = X^6 - 5$.
   (a) Give a set which generates K as an extension of $\mathbb{Q}$.
   (b) Give generators and relations for the group $G = \mathrm{Gal}(K/\mathbb{Q})$.
   (c) Indicate how the group generators from (b) act on the field generators from (a).

3. (a) Define what it means for a morphism in a category $\mathcal{C}$ to be a monomorphism.
   (b) Define what it means for a morphism in a category $\mathcal{C}$ to be an epimorphism.
   (c) Define what it means for a morphism in a category $\mathcal{C}$ to be an isomorphism.
   (d) Let $\mathcal{R}$ be the category of commutative rings with 1, with morphisms being unitary ring homomorphisms. Prove that the canonical map $i : \mathbb{Z} \to \mathbb{Q}$ is a monomorphism and an epimorphism in $\mathcal{R}$, but not an isomorphism.

4. Prove that the group G defined by the generators and relations

$$G = \langle x, y \mid x^4 = x^2y^{-2} = xy^{-1}xy = 1\rangle$$

   is isomorphic to the quaternion group Q_8.

5. Let R be a commutative ring with 1 and let I and J be ideals in R.
   (a) Prove that there exists a surjective R-module homomorphism $\phi : I \otimes_R J \to IJ$ such that $\phi(x \otimes y) = xy$ for all $x \in I$ and $y \in J$.
   (b) Give an example to show that the homomorphism ϕ does not have to be an isomorphism.
6. Let R be an integral domain and let M be a flat R-module. Prove that M is torsion-free as an R-module.
7. Let R be a commutative ring with 1, let J be the Jacobson radical of R, and let M be an R-module such that $JM = M$.
   (a) Prove that if M is finitely generated then $M = \{0\}$.
   (b) Give an example where M is not finitely generated and $M \neq \{0\}$. (Hint: Let R be a local ring.)
8. Let R be a unique factorization domain such that any two irreducible elements of R are associates. Prove that R is either a discrete valuation ring or a field.
9. Let R be a Dedekind domain and let I be a nonzero ideal in R. Prove that the quotient ring R/I has only finitely many ideals. Deduce that R/I is an Artinian ring.
10. Write down a representative for each isomorphism class of *noncommutative* semisimple rings with $512 = 2^9$ elements.
11. Let R be a ring with 1 such that every left R-module is free. Prove that R is a division ring. (Hint: The Wedderburn-Artin theorem is useful here.)