Ph.D. Exam – May 2014

Instructions: Do any 7 of the 11 problems, and mark clearly which problems you want graded if you try more than 7. State clearly any theorems that you use. Time: 4 hours.

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1. Suppose p_1, $p_2, \ldots, p_r$ are distinct primes, and $K = \mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_r})$, find the Galois group of $K/\mathbb{Q}$. How many fields F are there such that $\mathbb{Q} \subseteq F \subseteq K$?

2. (i) Show that if K/k and L/K are separable algebraic extensions, then so is L/k. (ii) Give an example of normal extensions K/k, L/K such that L/k is not normal.

3. Suppose K is an algebraic extension field of k and $\overline{k}$ is an algebraic closure of k. Show that K/k is separable if and only if the ring $\overline{k} \otimes_k K$ has no nonzero nilpotent elements.

4. Suppose A is a commutative ring with identity and $S \subset A$ is a multiplicative set not containing 0. (i) Show that there is a prime ideal $\mathfrak{p} \subset A$ such that $\mathfrak{p} \cap S = \phi$. (ii) Use part (i) to show that the intersection of all prime ideals of A is the set of nilpotent elements.

5. Suppose k is a field and A is a finitely generated commutative k-algebra. Let $\overline{k}$ be an algebraic closure of k. Note that the Galois group $G = \mathrm{Gal}(\overline{k}/k)$ acts on the set $\mathrm{Hom}_k(A, \overline{k})$ of k-algebra homomorphisms $A \to \overline{k}$ by composition. Construct a bijection of the set of orbits of G on $\mathrm{Hom}_k(A, \overline{k})$ with the set of maximal ideals in A.

6. Suppose A is an integral domain. Show that an ideal $I \subseteq A$ is invertible if and only if it is projective and finitely generated.

7. Suppose X is a nonempty set and $F(X)$ is the free group on X. Show that $F(X)$ is abelian if and only if X is a singleton.

8. (i) Define what it means for a group to be *nilpotent*. (ii) Show that if G is a finite nilpotent group, then any Sylow subgroup is normal.

9. Suppose A is a commutative ring with identity and B is a commutative A-algebra with structure map $f : A \to B$. Show that the forgetful functor from the category of B-modules to the category of A-modules induced by f has a left adjoint.

10. Let p be a prime. Find all the isomorphism classes of *noncommutative* semisimple rings A of cardinality p^{10}.

11. Suppose R is a finite commutative ring with identity, such that $x^3 = x$ for all $x \in R$. Show that R is a finite direct sum of fields isomorphic to $\mathbb{F}_2$ or $\mathbb{F}_3$.