Answer **seven** problems. (If you turn in more, the first seven will be graded.) Put your answers in numerical order and circle the numbers of the seven problems below your name. Within reason, you may quote theorems as long as you state them clearly.

Name: ______________________________

Problems to be graded: 1 2 3 4 5 6 7 8 9 10 11

Note. Below *ring* means associative ring with identity, and *module* means unital module.

1. (10 points) Let $f(x) = \sum_{i=0}^{n} a_i x^i \in \mathbf{Z}[x]$ be a polynomial with integer coefficients of degree $n > 1$. Suppose that for some k, with $0 < k < n$ and some prime p, we have $p \nmid a_n$; $p \nmid a_k$; $p \mid a_i$ for $i = 0, \ldots, k-1$; and $p^2 \nmid a_0$. Show that $f(x)$ has a factor of degree at least k which is irreducible in $\mathbf{Z}[x]$.

2. (10 points) Calculate the Galois group of $x^5 - 12x + 2$ over $\mathbf{Q}$, the field of rational numbers. Justify your answer carefully.

3. (a) (5 points) Give an example of fields $M \supset L \supset K$ such that M/L and L/K are normal extensions, but M/K is not normal. Justify your answer.

   (b) (5 points) Let $L \supset K$ be fields such that $[L : K] = 8$. Prove that there exist α, β, $\gamma \in L$ such that $L = K(\alpha, \beta, \gamma)$.

4. (10 points) Prove that the group defined by generators a and b and relations $a^2 = b^3 = 1$, $abab = 1$ is isomorphic to S_3.

5. (10 points) Let R be a ring with 1 and let

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

   be a short exact sequence of (left) R-modules. Show that for any (right) R-module M, the sequence

$$M \otimes_R A \xrightarrow{1_M \otimes f} M \otimes_R B \xrightarrow{1_M \otimes g} M \otimes_R C \to 0$$

   of tensor products, is exact.

6. (10 points) State and prove Hilbert's Basis Theorem.

7. (10 points) Prove the *Lying-Over Theorem*: Let S be an integral extension of an integral domain R, and let P be a prime ideal of R. Then, there exists a prime ideal Q of S, such that $Q \cap R = P$.

8. (10 points) Prove that a commutative ring R with identity is local if and only if, for all $r, s \in R$, $r + s = 1_R$ implies that either r or s is a unit.

9. Let R be a ring.
   (a) (3 points) Define what it means for an R-module M to be *projective*.
   (b) (4 points) Prove that any free R-module is projective.
   (c) (3 points) Prove that there exists some ring R and some projective R-module P such that P is a projective module but not a free module.

10. (10 points) Determine up to isomorphism all semisimple *noncommutative* rings with $512 = 2^9$ elements.

11. Let $\mathcal{C}$ be the category of all finitely generated $\mathbf{Z}$-modules, where $\mathbf{Z}$ is the ring of integers.
    (a) (3 points) Recall the definition of (direct) *product* in an arbitrary category $\mathcal{D}$.
    (b) (4 points) Prove from your definition that, given a finite set S of objects in $\mathcal{C}$, there is a product P of S in $\mathcal{C}$.
    (c) (3 points) Prove from your definition that there exists some set T of objects in $\mathcal{C}$, such that there is no product of T in $\mathcal{C}$.