Ph. D. Algebra Exam **May, 2013**

Answer seven problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Let K be a field and let $f(x) \in K[x]$ be a non-zero polynomial.
   (a) Define what is a *splitting field* for $f(x)$ over K.
   (b) If F_1 and F_2 are splitting fields for $f(x)$ over K prove, from your definition, that F_1 and F_2 are isomorphic fields.
2. Let $F = \mathbb{Q}\left(\sqrt{3}, \sqrt{5}\right)$ be the smallest subfield of $\mathbb{C}$ which contains $\sqrt{3}$ and $\sqrt{5}$. How many subfields does F have?
3. Let A and B be finite abelian groups and suppose that $|A|$ and $|B|$ are relatively prime. Let $T = A \otimes_{\mathbb{Z}} B$. Show that $|T| = 1$.
4. Let $\mathcal{C}$ be a category and let $(A_i)_{i \in I}$ be a family of objects of $\mathcal{C}$.
   (a) Define what is a *coproduct* of $(A_i)_{i \in I}$ in $\mathcal{C}$.
   (b) Prove that any two coproducts of $(A_i)_{i \in I}$ in $\mathcal{C}$ are *equivalent* (some texts would say *isomorphic*) in $\mathcal{C}$.
5. Let R be a ring.
   (a) Define what it means for an R-module M to be *projective.*
   (b) Prove that if
   $$0 \to A \to B \to P \to 0$$
   is an exact sequence of R-modules and P is projective then the exact sequence splits.
6. Let F be a free group with free generators a and b. Let N be the smallest normal subgroup of F such that $a^2 \in N$ and $b^2 \in N$. Prove that F/N is infinite.
7. State and prove the Lying-Over Theorem on prime ideals and ring extensions.

8. Prove the following version of Nakayama's Lemma. Let A be a commutative ring with identity, let M be a finitely generated A-module, and let I be an ideal contained in the intersection of all the maximal ideals of A. Show that if $IM = M$ then $M = \{0\}$.

9. An invertible ideal in an integral domain that is a local ring is principal.

10. Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$ and $v \in \overline{\mathbb{Q}}$ with $v \notin \mathbb{Q}$. Let E be a subfield of $\overline{\mathbb{Q}}$ maximal with respect to the condition that $v \notin E$. Prove that every finite-dimensional extension of E is cyclic.

11. Classify (up to ring isomorphism) all semisimple rings of order 720.