Department Of Mathematics PhD Algebra Examination

Answer **seven** problems. (If you turn in more, the first seven will be graded.) Put your answers in numerical order and circle the numbers of the seven problems below your name. Within reason, you may quote theorems as long as you state them clearly.

Name: ____________________

Problems to be graded: 1 2 3 4 5 6 7 8 9 10 11

1. (a) (5 points) Show that for all $n \geq 0$ there exists an irreducible polynomial $f \in \mathbb{Q}[x]$ of degree greater than n that does not satisfy the hypotheses of Eisenstein's criterion.

   (b) (5 points) Give an example, with proofs, of an irreducible polynomial which is not separable.

2. (10 points) Let $n \geq 1$ and let F be a splitting field of $x^n - 1 \in \mathbb{Q}[x]$.

   (a) (5 points) Prove that the Galois group of F over $\mathbb{Q}$ is abelian.

   (b) (5 points) When $n = 8$ show that the Galois group is of order 4 and not cyclic.

3. (10 points) Prove that the Galois group of $x^5 - 4x + 2 \in \mathbb{Q}[x]$ is the symmetric group of degree 5.

4. (10 points) Let F be a free group on a finite sets X of size n. Prove that the quotient of F by its commutator subgroup is isomorphic to $\mathbb{Z}^n$.

5. (10 points) Let R be a ring with 1 and let

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

   be a short exact sequence of (left) R-modules. Show that for any (right) R-module M, the sequence

$$M \otimes_R A \xrightarrow{1_M \otimes f} M \otimes_R B \xrightarrow{1_M \otimes g} M \otimes_R C \to 0$$

   of tensor products, is exact.

6. Let F be a field and $F[[x]]$ the ring of formal power series in the indeterminate x.

   (a) (4 points) Prove that $F[[x]]$ is a principal ideal domain.

   (b) (3 points) Determine all of the prime ideals of $F[[x]]$.

   (c) (3 points) Is $F[[x]]$ Noetherian? Artinian?

7. All parts involve ideals in a commutative ring with identity. Let S be a multiplicative set in R.
   (a) (2 points) Prove that if I is an ideal of R then $S^{-1}I = \{\frac{a}{s} \mid a \in I, s \in S\}$ is an ideal in the ring $S^{-1}R$, the localization of R with respect to S.
   (b) (3 points) Define the *radical* $\operatorname{Rad} I$ of an ideal I and show that it is an ideal.
   (c) (5 points) Prove that $S^{-1}(\operatorname{Rad} I) = \operatorname{Rad}(S^{-1}I)$.

8. You may not answer this problem just by quoting Nakayama's Lemma. Let R be a commutative local ring with 1 and J its unique maximal ideal.
   (a) (3 points) Prove that $1 - a$ is a unit for every element a of J,
   (b) (7 points) Let M be an R-module, with generators x_1, x_2, x_3 and suppose that x_2 and x_3 lie in JM. Show that $M = Rx_1$.

9. Let R be a ring with 1. All modules are unitary.
   (a) (2 points) Define the concept of a *projective* R-module.
   (b) (2 points) Show that if P is a projective R-module, then there is an R-module K such that $P \oplus K$ is isomorphic to a free module.
   (c) (3 points) Show that for every nonzero element x of a free $\mathbb{Z}$-module F there is a homomorphism from F to a finite group such that the image of x is nontrivial.
   (d) (3 points) Prove that $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module. (Hint: Show that Q has no nontrivial finite quotients. Then use the earlier parts.)

10. (10 points) Let D be a division ring with center K and suppose a and b are elements of D which are algebraic over K and have the same minimum polynomial. Prove that $b = dad^{-1}$ for some nonzero element $d \in D$.

11. Let $\mathcal{C}$ be a category and let $(A_i)_{i\in I}$ be a family of objects of $\mathcal{C}$.
    (a) (5 points) Define what is a *coproduct* of $(A_i)_{i\in I}$ in $\mathcal{C}$.
    (b) (5 points) Prove that in the category of sets the coproduct of two sets is their disjoint union.