Answer **seven** problems. (If you turn in more, the first seven will be graded.) Put your answers in numerical order and write the numbers of the seven problems on the front page. Within reason, you may quote theorems as long as you state them clearly.

Name: ______________________________

1. Let K be a field and let $f \in K[x]$
   (a) (4 points) State precisely a theorem concerning the existence and uniqueness of splitting fields for f over K.
   (b) (6 points) Prove that for any prime power p^n, there is, up to isomorphism a unique field of order p^n.
2. (10 points) Let F be a normal extension of K and $f \in K[x]$ an irreducible polynomial. Prove that in a factorization of f in $F[x]$ the irreducible factors all have the same degree.
3. (10 points) Let F be a free group on the set X. Show that the automorphism group of F has a subgroup isomorphic to the symmetric group of X.
4. (10 points) Let R be a ring with 1 and let

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

be a short exact sequence of (left) R-modules. Show that for any (left) R-module M, the sequence

$$0 \to \hom_R(M, A) \xrightarrow{\overline{f}} \hom_R(M, B) \xrightarrow{\overline{g}} \hom_R(M, C),$$

induced by composition of maps, is exact.

5. (10 points) Prove the Hilbert Basis Theorem: If R is a commutative Noetherian ring with identity then so is $R[x]$.
6. All parts involve ideals in a commutative ring with identity.
   (a) (2 points) Define the terms *primary ideal* and *radical* of an ideal.
   (b) (3 points) Prove that the radical of a primary ideal is prime.
   (c) (5 points) Is it true that an ideal is primary iff it is a power of a prime ideal? Answers without an explanation will receive no credit.
7. (For this question you **may not** quote Nakayama's Lemma.) Let R be a commutative local ring with 1 and J its unique maximal ideal.
   (a) (3 points) Show that $1 - j$ is a unit for every $j \in J$.

(b) (7 points) Show that if A is a finitely generated R-module such that $JA = A$, then $A = 0$. (Hint: Consider a generating set for A of minimal size and derive a contradiction.)

8. Let R be an integral domain with fraction field K. For $a \in K$, let $D(a) = \{r \in R \mid ra \in R\}$.

   (a) (2 points) Show that $D(a)$ is an ideal and that $D(a) = R$ iff $a \in R$.

   (b) (3 points) For each prime ideal P, let R_P denote the elements of K which can be represented as a fraction having denominator not in P. Show that $a \in R_P$ iff $D(a) \nsubseteq P$.

   (c) (5 points) Deduce that $\bigcap_P R_P = R$, where the intersection is taken over the maximal ideals of R.

9. Provide examples, with proof of:

   (a) (5 points) an integral domain which is not integrally closed in its field of fractions.

   (b) (5 points) a commutative ring with 1 that is not Noetherian.

10. (10 points) Classify (up to ring isomorphism) all semisimple rings of order 720.

11. Let $\mathcal{C}$ be a category and let $(A_i)_{i\in I}$ be a family of objects of $\mathcal{C}$.

    (a) (5 points) Define what is a *product* of $(A_i)_{i\in I}$ in $\mathcal{C}$.

    (b) (5 points) Prove that any two products of $(A_i)_{i\in I}$ in $\mathcal{C}$ are equivalent in $\mathcal{C}$.