Ph. D. Algebra Exam **May 12, 2011**

Answer seven problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Let K be a finite field, and let F be a subfield of K. Prove that the extension K/F is Galois.

2. (a) Determine a splitting field K for the polynomial $X^3 - 2 \in \mathbb{Q}[X]$.
   (b) Determine $\mathrm{Gal}(K/\mathbb{Q})$ and give an explicit description of the action of $\mathrm{Gal}(K/\mathbb{Q})$ on K.
   (c) For each subgroup H of $\mathrm{Gal}(K/\mathbb{Q})$ find the subfield of K fixed by H.

3. Let A and B be abelian groups, and $T = A \otimes_{\mathbb{Z}} B$. Let $\phi : A \to A$ be a group homomorphism. Prove that there is a group homomorphism
$$\psi : T \to T$$
such that, for all $a \in A$ and $b \in B$, we have
$$\psi(a \otimes_{\mathbb{Z}} b) = \phi(a) \otimes_{\mathbb{Z}} b.$$

4. (a) Let $\mathcal{C}$ be a concrete category and let S be a set. Define what it means to say that an object in $\mathcal{C}$ is *free* for S.
   (b) Prove that any two objects in $\mathcal{C}$ which are free for S are equivalent in $\mathcal{C}$.

5. Let R be a ring.
   (a) Define what it means for an R-module M to be *projective*.
   (b) Prove that any free R-module is projective.

6. Let F be a free group on a set S. Prove that F is abelian if and only if $|S| \leq 1$.

7. State and prove Hilbert's Basis Theorem.

8. Let R be a commutative ring with identity. Let S be a non-empty subset of R which is closed under multiplication and is such that $0 \notin S$. Prove that there exists a prime ideal P of R such that $P \cap S = \emptyset$.

9. Let R be a Dedekind domain, and let I be a non-zero ideal of R. Prove that R/I is Artinian.

10. (a) Let R be an integral domain. Define what is a *fractional ideal* of R.

    (b) Define what it means for a fractional ideal of R to be *invertible*.

    (c) Prove from your definition that if R is a principal ideal domain then every fractional ideal of R is invertible.

11. Let R and S be *non-commutative* semisimple rings with $|R| = |S| = 2^2 \cdot 3^4 \cdot 5$. Does it follow that R is isomorphic to S? Justify your answer.