Ph. D. Algebra Exam **May 18, 2010**

Answer seven problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Prove that every field has an algebraic closure.

2. (a) Determine a splitting field K for the polynomial $X^8 - 1 \in \mathbb{Q}[X]$.
   (b) Determine $\mathrm{Gal}(K/\mathbb{Q})$ and give an explicit description of the action of $\mathrm{Gal}(K/\mathbb{Q})$ on K.
   (c) For each subgroup H of $\mathrm{Gal}(K/\mathbb{Q})$ find the subfield of K fixed by H.

3. Let E and F be fields such that $\mathrm{char}(E) \neq \mathrm{char}(F)$. Let V be a vector space over E and let W be a vector space over F. Prove that $V \otimes_{\mathbb{Z}} W$ is trivial.

4. (a) Give the definition of a covariant functor from a category $\mathcal{C}$ to a category $\mathcal{D}$.
   (b) Give the definition of a concrete category.
   (c) Define what it means for an object F in a concrete category to be free on a set S.
   (d) Given a set S, construct a free object on S in the category of pointed sets.

5. (a) Define what it means for an R-module M to be injective.
   (b) Let $\{M_\lambda : \lambda \in \Lambda\}$ be a nonempty collection of injective R-modules. Prove that the product $\prod_{\lambda \in \Lambda} M_\lambda$ is injective.

6. Let F be a free group on a set S with $|S| > 1$. Prove that F is not solvable.

7. Let R be a commutative Noetherian ring with 1 and let $I \subset R$ be an ideal. Prove that I has a primary decomposition.

8. Let S be a commutative ring with 1, let R be a subring of S which contains 1_S, and let $x \in S$. Suppose there is a subring $T \subset S$ such that $T \supset R[x]$ and T is finitely generated as an R-module. Prove that x is integral over R.

9. Prove Noether's Normalization Lemma: Let k be a field and let A be a finitely generated k-algebra with 1. Then there are $y_1, \ldots, y_q \in A$ such that $y_1, \ldots, y_q$ are algebraically independent over k and A is integral over $k[y_1, \ldots, y_q]$.

10. (a) Prove that every nonzero proper ideal in a Dedekind domain is a product of prime ideals.

    (b) Give an example of an integral domain R and a nonzero proper ideal $I \subset R$ such that I cannot be expressed as a product of prime ideals.

11. Let G be a finite group of order n and let F be a field.

    (a) Prove that if $\mathrm{char}(F) \nmid n$ then for every $F[G]$-module V and every $F[G]$-submodule $W \subset V$ there is an $F[G]$-submodule $W' \subset V$ such that $V = W \oplus W'$.

    (b) Show that if $\mathrm{char}(F) \mid n$ then there exists an $F[G]$-module V and an $F[G]$-submodule $W \subset V$ such that there is no $F[G]$-submodule $W' \subset V$ such that $V = W \oplus W'$.