Ph.D. Algebra Exam – September 2009

Time allowed: Four Hours

Do **seven** of the following eleven problems. Do not turn in more than seven problems. You must show your work. Answers with no work and/or no explanations will receive **no** credit. State clearly any theorem you use in your proofs.

In the problems, $\mathbf{Z}$, resp. $\mathbf{Q}$, $\mathbf{C}$, is the set of all integers, resp. of all rational numbers, of all complex numbers.

1. State and prove the Jordan-Hölder theorem for finite groups.

2. (a) Show that the additive group of a field is *never* a free abelian group.
(b) Show that the multiplicative group of positive rational numbers is a free abelian group, and find a basis.

3. A category $\mathcal{C}$ is *small* if its class of objects is a set, and *skeletal* if for all X and Y in $\mathcal{C}$, $X \simeq Y$ implies $X = Y$. Show that any small category is equivalent to a skeletal one (choose an object from each isomorphism class).

4. Let R be a commutative ring. (a) Define what it means for an R-module M to be *flat*. (b) Show that the polynomial ring $R[x]$, considered as an R-module in the natural way, is flat.

5. Show that if R is a commutative ring with identity and every prime ideal of R is finitely generated, then R is Noetherian.

6. Show that for any extension $A \to B$ of commutative rings, any A-module M and any B-module N, there is an isomorphism

$$\mathrm{Hom}_B(M \otimes_A B, N) \simeq \mathrm{Hom}_A(M, N_A)$$

functorial in M and N (here N_A denotes N considered as an A-module by means of the ring homomorphism $A \to B$).

7. (a) Show that if L/F, K/L are finite separable extensions, then so is K/F. (b) Do the same for purely inseparable extensions.

8. (a) Show that if K/F is a normal extension and $F \subseteq L \subseteq K$ is an intermediate field, then K/L is normal. (b) Give an example where L/F is not normal. (c) Give an example of normal extensions L/F, K/L where K/F is *not* normal.

9. Let p is a prime and K is the splitting field over $\mathbf{Q}$ of $X^p - 1$. (a) Show that K contains a unique quadratic extension of $\mathbf{Q}$. (b) When is this quadratic extension real? (You may assume as known the structure of the Galois group of $K/\mathbf{Q}$. hint: if ζ is primitive p^{th} root of unity, complex conjugation sends $\zeta \mapsto \zeta^{-1}$).

10. Let k be a field and R a finite-dimensional k-algebra. Show that if R is semisimple as a ring, it is a direct sum of extensions fields of k.

11. Classify up to isomorphism all semisimple rings with 720 elements.